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Probabilistic Engineering Mechanics 17 (2002) 265–272

PROBABILISTIC ENGINEERING MECHANICS

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Spectral density estimation of stochastic vector processes

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Abstract

A spectral density matrix estimator for stationary stochastic vector processes is studied. As the duration of the analyzed data tends to infinity, the probability distribution for this estimator at each frequency approaches a complex Wishart distribution with mean equal to an aliased version of the power spectral density at that frequency. It is shown that the spectral density matrix estimators corresponding to different frequencies are asymptotically statistically independent. These properties hold for general stationary vector processes, not only Gaussian processes, and they allow efficient calculation of updated probabilities when formulating a Bayesian model updating problem in the frequency domain using response data. A three-degree-of-freedom Duffing oscillator is used to verify the results. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Stochastic vector process; Spectral density estimator; Aliasing; FFT; Stationary process

1. Introduction

Frequency-domain analysis is widely used in almost every field of research involving analysis of continuous or discrete time-varying signals. In particular, Fourier analysis is a very popular tool for time-history data processing because it provides an efficient and systematic procedure to capture important frequency-content information. A common task is to fit a parameterized theoretical Fourier spectrum to the spectrum obtained from measured time histories. A straightforward least-squares fit is usually found to give poor results. Therefore, it is desirable to develop a theoretically sound approach for fitting spectra. In this paper, statistical properties of a spectral density estimator, such as the mean and the covariance matrix of the estimator, are presented for each frequency. Furthermore, it is shown that the spectral density estimator is asymptotically statistically independent at different frequencies for general stationary vector processes. The proof does not assume a Gaussian stochastic model for the signal process. A three-degree-of-freedom Duffing oscillator is used to verify the results. These results allow for a mathematically rigorous frequency-domain formulation of the problem of model updating within a Bayesian framework [1,2] using response data. The approach is applicable in the case of a stationary stochastic model for the response of linear or non-linear systems subject to Gaussian or non-Gaussian input.

2. Spectral density estimator

Consider a process generating time-history signals that are modeled as realizations of a real continuous-time stationary stochastic vector process $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_d(t)]^{\mathrm{T}}$, that may be Gaussian or non-Gaussian and that has mean $E[\mathbf{x}(t)] \equiv \boldsymbol{\mu} = [\mu_1, \mu_2, \ldots, \mu_d]^{\mathrm{T}}$, $\forall t \in \mathbb{R}$. Now, consider the corresponding discrete-time stochastic vector process $\{\mathbf{x}(0), \mathbf{x}(\Delta t), \ldots, \ldots, \mathbf{x}((N-1)\Delta t)\}$ whose realization corresponds to the sampled data in applications. We define a related frequency-domain stochastic vector process $\mathbf{X}^N$ with its αth element at frequency ω_k given by a modified discrete Fourier transform

$$X_\alpha^N(\omega_k) \equiv \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}\omega_k j\Delta t} x_\alpha(j\Delta t) \tag{1}$$

where $\omega_k = k\Delta\omega$, $k = 0, 1, \ldots, \mathrm{INT}(N/2)$ and $\Delta\omega = 2\pi/N\Delta t$.

Furthermore, denote the real and imaginary parts of $X_\alpha^N(\omega_k)$ by $R_\alpha^N(\omega_k)$ and $I_\alpha^N(\omega_k)$, respectively, that is:

$$R_\alpha^N(\omega_k) = \mathrm{Re}\Big[X_\alpha^N(\omega_k)\Big], \qquad I_\alpha^N(\omega_k) = \mathrm{Im}\Big[X_\alpha^N(\omega_k)\Big] \tag{2}$$

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PII: S0266-8920(02)00011-5

Note that

$$E\Big[X_\alpha^N(\omega_k)\Big] = \frac{\mu_\alpha}{\sqrt{N}} \sum_{j=0}^{N-1} \mathrm{e}^{-\mathrm{i}\omega_k j\Delta t} = \frac{\mu_\alpha(1-\mathrm{e}^{-\mathrm{i}\omega_k N\Delta t})}{\sqrt{N}(1-\mathrm{e}^{-\mathrm{i}\omega_k\Delta t})}$$

$$= \frac{\mu_\alpha(1-\mathrm{e}^{-2k\pi\mathrm{i}})}{\sqrt{N}(1-\mathrm{e}^{-\mathrm{i}\omega_k\Delta t})} = 0$$

Therefore, $E[R_\alpha^N(\omega_k)] = 0$ and $E[I_\alpha^N(\omega_k)] = 0$.

The (α, β) element of the spectral density matrix estimator is defined for $\alpha, \beta = 1, 2, \ldots, d$ by

$$S_{\alpha,\beta}^N(\omega_k) \equiv \frac{\Delta t}{2\pi} X_\alpha^N(\omega_k) X_\beta^N(\omega_k)^*$$

$$\equiv \frac{\Delta t}{2\pi N} \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} \mathrm{e}^{-\mathrm{i}\omega_k(j-l)\Delta t} x_\alpha(j\Delta t) x_\beta(l\Delta t) \tag{3}$$

where z^* denotes the complex conjugate of z.

In Section 3, some important statistical properties of the spectral density estimator are introduced. In Section 4, it is shown that $\mathbf{S}^N(\omega)$ and $\mathbf{S}^N(\omega')$ are independent asymptotically as $N \to \infty$, where $\omega \neq \omega'$ and $0 < \omega, \omega' < \pi/\Delta t$, the Nyquist frequency.

3. Statistical properties of the spectral density estimator

In this section, the probability density function of the spectral density estimator at a particular frequency ω_k is presented.

First, by taking expectation of Eq. (3), one obtains the following

$$E\Big[S_{\alpha,\beta}^N(\omega_k)\Big] = \frac{\Delta t}{2\pi N} \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} \mathrm{e}^{-\mathrm{i}\omega_k(j-l)\Delta t} \phi_{\alpha,\beta}[(j-l)\Delta t] \tag{4}$$

where $\phi_{\alpha,\beta}[(j-l)\Delta t] \equiv E[x_\alpha(j\Delta t)x_\beta(l\Delta t)] - \mu_\alpha\mu_\beta$, for $\alpha, \beta = 1, 2, \ldots, d$, is the cross-covariance function between x_α and x_β with time lag $(j-l)\Delta t$.

By grouping together terms with the same values of $(j-l)\Delta t$, one can obtain the following expression for the expected value of the spectral density estimator

$$E[S_{\alpha,\beta}^N(\omega_k)]$$

$$= \frac{\Delta t}{4\pi} \sum_{j=0}^{N-1} c_j \Big[\phi_{\alpha,\beta}(j\Delta t)\mathrm{e}^{-\mathrm{i}\omega_k j\Delta t} + \phi_{\alpha,\beta}(-j\Delta t)\mathrm{e}^{\mathrm{i}\omega_k j\Delta t}\Big] \tag{5}$$

where c_j is given by:

$$c_0 = 1, \qquad c_j = 2\left(1 - \frac{j}{N}\right), \; j \geq 1 \tag{6}$$

Note that this relationship is exact and takes care of the aliasing and leakage effects automatically. Furthermore, $E[S_{\alpha,\beta}^N(\omega_k)]$ can be calculated efficiently using the function FFT in Matlab [3].

By using $\phi_{\alpha,\beta}(\tau) = \int_{-\infty}^{\infty} S_{\alpha,\beta}(\Omega)\mathrm{e}^{\mathrm{i}\Omega\tau}\,\mathrm{d}\Omega$, Eq. (4) can be rewritten as

$$E\Big[S_{\alpha,\beta}^N(\omega_k)\Big]$$

$$= \frac{\Delta t}{2\pi N} \int_{-\infty}^{\infty} \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} \mathrm{e}^{\mathrm{i}(\Omega-\omega_k)(j-l)\Delta t} S_{\alpha,\beta}(\Omega)\,\mathrm{d}\Omega$$

$$= \int_{-\infty}^{\infty} \sum_{j=0}^{N-1} \sum_{l=0}^{N-1} \frac{1}{2\pi N} \mathrm{e}^{\mathrm{i}(\xi-\omega_k\Delta t)(j-l)} S_{\alpha,\beta}\left(\frac{\xi}{\Delta t}\right)\mathrm{d}\xi$$

$$= \int_{-\infty}^{\infty} F^N(\xi - \omega_k\Delta t) S_{\alpha,\beta}\left(\frac{\xi}{\Delta t}\right)\mathrm{d}\xi \tag{7}$$

where $\xi \equiv \Omega\Delta t$ and $F^N(\eta)$ is the Fejer kernel [4]:

$$F^N(\eta) = \frac{\sin^2(N\eta/2)}{2\pi N \sin^2(\eta/2)} \tag{8}$$

Note that $F^N(\eta)$ is periodic with period 2π. By taking $N \to \infty$

$$\lim_{N\to\infty} E\Big[S_{\alpha,\beta}^N(\omega_k)\Big]$$

$$= \int_{-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \delta(\xi - \omega_k\Delta t + 2j\pi) S_{\alpha,\beta}\left(\frac{\xi}{\Delta t}\right)\mathrm{d}\xi$$

$$= S_{\alpha,\beta}(\omega_k) + \sum_{j=1}^{\infty}\left[S_{\alpha,\beta}\left(\omega_k + \frac{2j\pi}{\Delta t}\right) + S_{\alpha,\beta}\left(\omega_k - \frac{2j\pi}{\Delta t}\right)\right]$$

$$= S_{\alpha,\beta}(\omega_k) + \sum_{j=1}^{\infty}\left[S_{\alpha,\beta}\left(\frac{2j\pi}{\Delta t} + \omega_k\right) + S_{\alpha,\beta}\left(\frac{2j\pi}{\Delta t} - \omega_k\right)^*\right] \tag{9}$$

where the terms in the summation produce an aliased version of the power spectral density at frequency ω_k.

Now, $R_\alpha^N(\omega_k)$ and $I_\alpha^N(\omega_k)$ are Gaussian distributed as $N \to \infty$ by using the Central Limit Theorem [5]. Furthermore, as $N \to \infty$, it was shown [6] that the covariance matrix $\boldsymbol{\Gamma}_X(\omega_k)$ of the vector $[\mathbf{R}^N(\omega_k)^{\mathrm{T}}, \mathbf{I}^N(\omega_k)^{\mathrm{T}}]^{\mathrm{T}}$ has the form:

$$\lim_{N\to\infty} \boldsymbol{\Gamma}_X(\omega_k) = \begin{bmatrix} \frac{1}{2}\mathrm{Re}\Big\{E\Big[\mathbf{S}^N(\omega_k)\Big]\Big\} & -\frac{1}{2}\mathrm{Im}\Big\{E\Big[\mathbf{S}^N(\omega_k)\Big]\Big\} \\ \frac{1}{2}\mathrm{Im}\Big\{E\Big[\mathbf{S}^N(\omega_k)\Big]\Big\} & \frac{1}{2}\mathrm{Re}\Big\{E\Big[\mathbf{S}^N(\omega_k)\Big]\Big\} \end{bmatrix} \tag{10}$$

Eq. (10) states that $\mathbf{R}^N(\omega_k)$ and $\mathbf{I}^N(\omega_k)$ have equal covariance matrices $(1/2)\mathrm{Re}\{E[\mathbf{S}^N(\omega_k)]\}$, as $N \to \infty$. Also, as $N \to \infty$, the cross-covariance between $\mathbf{R}^N(\omega_k)$ and $\mathbf{I}^N(\omega_k)$ has the property $-\mathrm{Im}\{E[\mathbf{S}^N(\omega_k)]\} = \mathrm{Im}\{E[\mathbf{S}^N(\omega_k)]\}^{\mathrm{T}}$, i.e.

$$\lim_{N\to\infty} E\Big[R_\alpha^N(\omega_k) I_\beta^N(\omega_k)\Big] = -\lim_{N\to\infty} E\Big[I_\alpha^N(\omega_k) R_\beta^N(\omega_k)\Big]$$

because of the symmetry of $\boldsymbol{\Gamma}_X(\omega_k)$. The latter property implies also that the diagonal elements of $\mathrm{Im}\{E[\mathbf{S}^N(\omega_k)]\}$ are equal to zero as $N \to \infty$, i.e.

$$\lim_{N\to\infty} E[R_\alpha^N(\omega_k) I_\alpha^N(\omega_k)] = 0, \qquad \alpha = 1,2,\dots,d.$$

Therefore, the complex vector $\mathbf{X}^N(\omega_k)$ has a complex multivariate Gaussian distribution [7] with zero mean as $N \to \infty$.

Assume now that there is a set of M independent and identically distributed time histories whose realizations correspond to sampled data from the same process $\mathbf{x}(t)$. As $N \to \infty$, the corresponding frequency-domain stochastic processes $\mathbf{X}^{N,(m)}(\omega_k)$, $m = 1,2,\dots,M$, are independent and follow an identical complex d-variate Gaussian distribution with zero mean [7]. Also, as $N \to \infty$ and if $M \geq d$, the average spectral density estimator

$$\mathbf{S}^{N,M}(\omega_k) = \frac{1}{M}\sum_{m=1}^{M} \mathbf{S}^{N,(m)}(\omega_k) \tag{11}$$

follows a central complex Wishart distribution of dimension d with M degrees-of-freedom [7,8] and mean $E[\mathbf{S}^{N,M}(\omega_k)] = E[\mathbf{S}^N(\omega_k)]$ given by Eq. (9):

$$p\left[\mathbf{S}^{N,M}(\omega_k)\right] = \frac{\pi^{-d(d-1)/2} M^{M-d+d^2} \left|\mathbf{S}^{N,M}(\omega_k)\right|^{M-d}}{\left[\prod_{p=1}^{d}(M-p)!\right]\left|E\left[\mathbf{S}^N(\omega_k)\right]\right|^{M}} \times \exp\left(-M\,\mathrm{tr}\left\{E\left[\mathbf{S}^N(\omega_k)\right]^{-1}\mathbf{S}^{N,M}(\omega_k)\right\}\right) \tag{12}$$

where the (α,β) element of the matrix $E[\mathbf{S}^N(\omega_k)]$ is given by Eq. (5). Here, $|\mathbf{A}|$ and $\mathrm{tr}[\mathbf{A}]$ denote the determinant and the trace, respectively, of a matrix $\mathbf{A}$.

Note that in the special case of $d = 1$, this distribution reduces to a Chi-square distribution with $2M$ degrees-of-freedom and mean $E[S^N(\omega_k)]$, which is given by:

$$p\left[S^{N,M}(\omega_k)\right] = \frac{M^M\left[S^{N,M}(\omega_k)\right]^{M-1}}{(M-1)!\{E[S^N(\omega_k)]\}^M}\exp\left(-\frac{MS^{N,M}(\omega_k)}{E[S^N(\omega_k)]}\right) \tag{13}$$

Another special case is when $M = 1$ (i.e. no averaging is performed), then each of the diagonal elements $S_{\alpha,\alpha}^N(\omega_k)$, $\alpha = 1,2,\dots,d$ follows an exponential distribution as $N \to \infty$:

$$p\left[S_{\alpha,\alpha}^N(\omega_k)\right] = \frac{1}{E[S_{\alpha,\alpha}^N(\omega_k)]}\exp\left(-\frac{S_{\alpha,\alpha}^N(\omega_k)}{E[S_{\alpha,\alpha}^N(\omega_k)]}\right) \tag{14}$$

In this section, the PDF of the spectral density matrix estimator was given at a particular frequency. In Section 4, it is shown that the spectral density matrix estimators are independent at different frequencies.

4. Asymptotic independence properties of the spectral density estimator

In this section, it is shown that the spectral density estimators are independent at any two different frequencies ω and ω' as $N \to \infty$.

4.1. $R_\alpha^N(\omega)$ with $R_\beta^N(\omega')$

First, by using Eqs. (1) and (2), one can easily obtain the following:

$$\begin{aligned} &E\left[\left(R_\alpha^N(\omega) - E\left[R_\alpha^N(\omega)\right]\right)\left(R_\beta^N(\omega') - E\left[R_\beta^N(\omega')\right]\right)\right] \\ &= E\left[R_\alpha^N(\omega)R_\beta^N(\omega')\right] - E\left[R_\alpha^N(\omega)\right]E\left[R_\beta^N(\omega')\right] \\ &= \frac{1}{N}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1}(E[x_\alpha(j\Delta t)x_\beta(l\Delta t)] \\ &\quad - \mu_\alpha\mu_\beta)\cos(j\omega\Delta t)\cos(l\omega'\Delta t) \end{aligned} \tag{15}$$

Let $S_{\alpha,\beta}(\Omega)$, which is assumed to be finite $\forall \Omega \in \mathbb{R}$, be the cross-spectral density between x_α and x_β at frequency Ω. By using the fact that the cross-covariance function $\phi_{\alpha,\beta}(\tau) = \int_{-\infty}^{\infty} S_{\alpha,\beta}(\Omega)\mathrm{e}^{\mathrm{i}\Omega\tau}\,\mathrm{d}\Omega$ and $\cos(z) = (\mathrm{e}^{\mathrm{i}z} + \mathrm{e}^{-\mathrm{i}z})/2$,

$$\begin{aligned} &E\left[\left(R_\alpha^N(\omega) - E\left[R_\alpha^N(\omega)\right]\right)\left(R_\beta^N(\omega') - E\left[R_\beta^N(\omega')\right]\right)\right] \\ &= \int_{-\infty}^{\infty}\frac{1}{4N}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1}\mathrm{e}^{\mathrm{i}\Omega(j-l)\Delta t}S_{\alpha,\beta}(\Omega)[\mathrm{e}^{\mathrm{i}j\omega\Delta t} + \mathrm{e}^{-\mathrm{i}j\omega\Delta t}] \\ &\quad \times[\mathrm{e}^{\mathrm{i}l\omega'\Delta t} + \mathrm{e}^{-\mathrm{i}l\omega'\Delta t}]\,\mathrm{d}\Omega \\ &= \int_{-\infty}^{\infty}\frac{1}{4N}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1}[\mathrm{e}^{\mathrm{i}j(\Omega+\omega)\Delta t + \mathrm{i}l(\omega'-\Omega)\Delta t} \\ &\quad + \mathrm{e}^{\mathrm{i}j(\Omega+\omega)\Delta t + \mathrm{i}l(-\omega'-\Omega)\Delta t} + \mathrm{e}^{\mathrm{i}j(\Omega-\omega)\Delta t + \mathrm{i}l(\omega'-\Omega)\Delta t} \\ &\quad + \mathrm{e}^{\mathrm{i}j(\Omega-\omega)\Delta t + \mathrm{i}l(-\omega'-\Omega)\Delta t}]S_{\alpha,\beta}(\Omega)\,\mathrm{d}\Omega \\ &= \int_{-\infty}^{\infty}H_{\alpha,\beta}^N(\Omega;\omega,\omega')\,\mathrm{d}\Omega + \int_{-\infty}^{\infty}H_{\alpha,\beta}^N(\Omega;\omega,-\omega')\,\mathrm{d}\Omega \\ &\quad + \int_{-\infty}^{\infty}H_{\alpha,\beta}^N(\Omega;-\omega,\omega')\,\mathrm{d}\Omega \\ &\quad + \int_{-\infty}^{\infty}H_{\alpha,\beta}^N(\Omega;-\omega,-\omega')\,\mathrm{d}\Omega \end{aligned} \tag{16}$$

where $H^N(\Omega;\omega,\omega')$ is defined as:

$$H_{\alpha,\beta}^N(\Omega;\omega,\omega') \equiv \frac{S_{\alpha,\beta}(\Omega)}{4N}\sum_{j=0}^{N-1}\mathrm{e}^{\mathrm{i}j(\Omega+\omega)\Delta t}\sum_{l=0}^{N-1}\mathrm{e}^{\mathrm{i}l(\omega'-\Omega)\Delta t} \tag{17}$$

If $|\Omega| \neq \omega$ and $|\Omega| \neq \omega'$

$$H_{\alpha,\beta}^N(\Omega;\omega,\omega') = \frac{[1-\mathrm{e}^{\mathrm{i}N\Delta t(\Omega+\omega)}][1-\mathrm{e}^{\mathrm{i}N\Delta t(\omega'-\Omega)}]}{4N[1-\mathrm{e}^{\mathrm{i}\Delta t(\Omega+\omega)}][1-\mathrm{e}^{\mathrm{i}\Delta t(\omega'-\Omega)}]}S_{\alpha,\beta}(\Omega) \tag{18}$$

By using $\sin(z) = (e^{iz} - e^{-iz})/2i$,

$$H^N_{\alpha,\beta}(\Omega;\omega,\omega')$$

$$= \frac{e^{i(N-1)\Delta t(\omega+\omega')/2}\sin\left[\frac{(\Omega+\omega)N\Delta t}{2}\right]\sin\left[\frac{(\omega'-\Omega)N\Delta t}{2}\right]}{4N\sin\left[\frac{(\Omega+\omega)\Delta t}{2}\right]\sin\left[\frac{(\omega'-\Omega)\Delta t}{2}\right]}S_{\alpha,\beta}(\Omega) \tag{19}$$

Then, the following inequality can be obtained since $|e^{ir}| = 1$ and $|\sin(r)| \le 1$ $(r \in \mathbb{R})$:

$$\left|H^N_{\alpha,\beta}(\Omega;\omega,\omega')\right| \le \left|\frac{S_{\alpha,\beta}(\Omega)}{4N\sin\left[\frac{(\Omega+\omega)\Delta t}{2}\right]\sin\left[\frac{(\omega'-\Omega)\Delta t}{2}\right]}\right| \tag{20}$$

By taking the limit as $N \to \infty$

$$\lim_{N\to\infty} H^N_{\alpha,\beta}(\Omega;\omega,\omega') = 0 \qquad \text{if } |\Omega| \neq \omega \text{ and } |\Omega| \neq \omega' \tag{21}$$

Similarly, $H^N_{\alpha,\beta}(\Omega;-\omega,\omega')$, $H^N_{\alpha,\beta}(\Omega;\omega,-\omega')$ and $H^N_{\alpha,\beta}(\Omega;-\omega,-\omega')$ also tend to zero as $N \to \infty$, if $|\Omega| \neq \omega$ and $|\Omega| \neq \omega'$.

Now, we consider $H^N_{\alpha,\beta}(\Omega;\omega,\omega')$ at $|\Omega| = \omega$ or $|\Omega| = \omega'$. First, at $\Omega = -\omega$

$$H^N_{\alpha,\beta}(\Omega;\omega,\omega') = \frac{S_{\alpha,\beta}(\Omega)}{4}\sum_{l=0}^{N-1} e^{il(\omega'-\Omega)\Delta t}$$

$$= \frac{S_{\alpha,\beta}(\Omega)[1 - e^{iN\Delta t(\omega'-\Omega)}]}{4[1 - e^{i\Delta t(\omega'-\Omega)}]} \tag{22}$$

which is finite. Similarly, it can be shown that $H^N_{\alpha,\beta}(\Omega;\omega,\omega')$ is finite at $\Omega = \omega'$. Next, consider $\Omega = \omega$ or $\Omega = -\omega'$

$$H^N_{\alpha,\beta}(\Omega;\omega,\omega') = \frac{[1 - e^{iN\Delta t(\Omega+\omega)}][1 - e^{iN\Delta t(\omega'-\Omega)}]}{4N[1 - e^{i\Delta t(\Omega+\omega)}][1 - e^{i\Delta t(\omega'-\Omega)}]}S_{\alpha,\beta}(\Omega) \tag{23}$$

As $N \to \infty$ and $\Omega = \omega$ or $-\omega'$,

$$\lim_{N\to\infty} H^N_{\alpha,\beta}(\Omega;\omega,\omega') = 0 \tag{24}$$

That is, $H^N_{\alpha,\beta}(\Omega;\omega,\omega')$ is finite at $\Omega = \pm\omega$ and $\Omega = \pm\omega'$. Similarly, it can be shown that $H^N_{\alpha,\beta}(\Omega;\omega,-\omega')$, $H^N_{\alpha,\beta}(\Omega;-\omega,\omega')$ and $H^N_{\alpha,\beta}(\Omega;-\omega,-\omega')$ are finite at $\Omega = \pm\omega$ and $\Omega = \pm\omega'$.

Therefore, $H^N_{\alpha,\beta}(\Omega;\omega,\omega')$, $H^N_{\alpha,\beta}(\Omega;\omega,-\omega')$, $H^N_{\alpha,\beta}(\Omega;-\omega,\omega')$ and $H^N_{\alpha,\beta}(\Omega;-\omega,-\omega')$ tend to zero as $N \to \infty$ if $|\Omega| \neq \omega$ and $|\Omega| \neq \omega'$ and they are finite as $N \to \infty$ if $|\Omega| = \omega$ or $|\Omega| = \omega'$. It can be concluded that $\int_{-\infty}^{\infty} H^N_{\alpha,\beta}(\Omega;\omega,\omega')\,d\Omega$, $\int_{-\infty}^{\infty} H^N_{\alpha,\beta}(\Omega;\omega,-\omega')\,d\Omega$, $\int_{-\infty}^{\infty} H^N_{\alpha,\beta}(\Omega;-\omega,\omega')\,d\Omega$ and $\int_{-\infty}^{\infty} H^N_{\alpha,\beta}(\Omega;-\omega,-\omega')\,d\Omega$ tend to zero as $N \to \infty$ and so from Eq. (16):

$$\lim_{N\to\infty} E\Big[\Big(R^N_\alpha(\omega) - E\Big[R^N_\alpha(\omega)\Big]\Big)\Big(R^N_\beta(\omega') - E\Big[R^N_\beta(\omega')\Big]\Big)\Big]$$
$$= 0, \text{ if } \omega \neq \omega' \tag{25}$$

4.2. $I^N_\alpha(\omega)$ with $I^N_\beta(\omega')$

Similarly, it can be proved that $\lim_{N\to\infty} E[(I^N_\alpha(\omega) - E[I^N_\alpha(\omega)])(I^N_\beta(\omega') - E[I^N_\beta(\omega')])] = 0$ as follows

$$E\Big[\Big(I^N_\alpha(\omega) - E\Big[I^N_\alpha(\omega)\Big]\Big)\Big(I^N_\beta(\omega') - E\Big[I^N_\beta(\omega')\Big]\Big)\Big]$$
$$= \frac{1}{N}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1}(E[x_\alpha(j\Delta t)x_\beta(l\Delta t)]$$
$$- \mu_\alpha\mu_\beta)\sin(j\omega\Delta t)\sin(l\omega'\Delta t)$$
$$= -\frac{1}{4N}\int_{-\infty}^{\infty}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1} e^{i\Omega(j-l)\Delta t}S_{\alpha,\beta}(\Omega)$$
$$\times[e^{ij\omega\Delta t} - e^{-ij\omega\Delta t}][e^{il\omega'\Delta t} - e^{-il\omega'\Delta t}]\,d\Omega$$
$$= -\int_{-\infty}^{\infty}\frac{1}{4N}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1}[e^{ij(\Omega+\omega)\Delta t + il(\omega'-\Omega)\Delta t}$$
$$- e^{ij(\Omega+\omega)\Delta t + il(-\omega'-\Omega)\Delta t} - e^{ij(\Omega-\omega)\Delta t + il(\omega'-\Omega)\Delta t}$$
$$+ e^{ij(\Omega-\omega)\Delta t + il(-\omega'-\Omega)\Delta t}]S_{\alpha,\beta}(\Omega)\,d\Omega$$
$$= -\int_{-\infty}^{\infty}\frac{1}{4N}\Bigg[\sum_{j=0}^{N-1} e^{ij(\Omega+\omega)\Delta t}\sum_{l=0}^{N-1} e^{il(\omega'-\Omega)\Delta t}$$
$$- \sum_{j=0}^{N-1} e^{ij(\Omega+\omega)\Delta t}\sum_{l=0}^{N-1} e^{il(-\omega'-\Omega)\Delta t}$$
$$- \sum_{j=0}^{N-1} e^{ij(\Omega-\omega)\Delta t}\sum_{l=0}^{N-1} e^{il(\omega'-\Omega)\Delta t}$$
$$+ \sum_{j=0}^{N-1} e^{ij(\Omega-\omega)\Delta t}\sum_{l=0}^{N-1} e^{il(-\omega'-\Omega)\Delta t}\Bigg]S_{\alpha,\beta}(\Omega)\,d\Omega$$
$$= -\int_{-\infty}^{\infty} H^N_{\alpha,\beta}(\Omega;\omega,\omega')\,d\Omega$$
$$+ \int_{-\infty}^{\infty} H^N_{\alpha,\beta}(\Omega;\omega,-\omega')\,d\Omega$$
$$+ \int_{-\infty}^{\infty} H^N_{\alpha,\beta}(\Omega;-\omega,\omega')\,d\Omega$$
$$- \int_{-\infty}^{\infty} H^N_{\alpha,\beta}(\Omega;-\omega,-\omega')\,d\Omega \tag{26}$$

where $H^N_{\alpha,\beta}$ is given by Eq. (17).

As shown in Section 4.1, $\int_{-\infty}^{\infty} H^N_{\alpha,\beta}(\Omega;\omega,\omega')\,d\Omega$, $\int_{-\infty}^{\infty} H^N_{\alpha,\beta}(\Omega;\omega,-\omega')\,d\Omega$, $\int_{-\infty}^{\infty} H^N_{\alpha,\beta}(\Omega;-\omega,\omega')\,d\Omega$ and $\int_{-\infty}^{\infty} H^N_{\alpha,\beta}(\Omega;-\omega,-\omega')\,d\Omega$ tend to zero as $N \to \infty$, so

$$\lim_{N\to\infty} E\Big[\Big(I^N_\alpha(\omega) - E\Big[I^N_\alpha(\omega)\Big]\Big)\Big(I^N_\beta(\omega') - E\Big[I^N_\beta(\omega')\Big]\Big)\Big] = 0,$$
$$\text{if } \omega \neq \omega' \tag{27}$$

4.3. $R^N_\alpha(\omega)$ with $I^N_\beta(\omega')$

Finally, we prove: $\lim_{N\to\infty} E[(R^N_\alpha(\omega) - E[R^N_\alpha(\omega)]) \times (I^N_\beta(\omega') - E[I^N_\beta(\omega')])] = 0$

$$E\left[\left(R^N_\alpha(\omega) - E\left[R^N_\alpha(\omega)\right]\right)\left(I^N_\beta(\omega') - E\left[I^N_\beta(\omega')\right]\right)\right]$$

$$= \frac{1}{N}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1}(E[x_\alpha(j\Delta t)x_\beta(l\Delta t)]$$

$$- \mu_\alpha\mu_\beta)\cos(j\omega\Delta t)\sin(l\omega'\Delta t)$$

$$= \int_{-\infty}^{\infty}\frac{1}{4N\mathrm{i}}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1}\mathrm{e}^{\mathrm{i}\Omega(j-l)\Delta t}S_{\alpha,\beta}(\Omega)[\mathrm{e}^{\mathrm{i}j\omega\Delta t} + \mathrm{e}^{-\mathrm{i}j\omega\Delta t}]$$

$$\times[\mathrm{e}^{\mathrm{i}l\omega'\Delta t} - \mathrm{e}^{-\mathrm{i}l\omega'\Delta t}]\,\mathrm{d}\Omega$$

$$= \int_{-\infty}^{\infty}\frac{1}{4N\mathrm{i}}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1}[\mathrm{e}^{\mathrm{i}j(\Omega+\omega)\Delta t+\mathrm{i}l(\omega'-\Omega)\Delta t}$$

$$- \mathrm{e}^{\mathrm{i}j(\Omega+\omega)\Delta t+\mathrm{i}l(-\omega'-\Omega)\Delta t} + \mathrm{e}^{\mathrm{i}j(\Omega-\omega)\Delta t+\mathrm{i}l(\omega'-\Omega)\Delta t}$$

$$- \mathrm{e}^{\mathrm{i}j(\Omega-\omega)\Delta t+\mathrm{i}l(-\omega'-\Omega)\Delta t}]S_{\alpha,\beta}(\Omega)\,\mathrm{d}\Omega$$

$$= \int_{-\infty}^{\infty}\frac{1}{4N\mathrm{i}}\left[\sum_{j=0}^{N-1}\mathrm{e}^{\mathrm{i}j(\Omega+\omega)\Delta t}\sum_{l=0}^{N-1}\mathrm{e}^{\mathrm{i}l(\omega'-\Omega)\Delta t}\right.$$

$$- \sum_{j=0}^{N-1}\mathrm{e}^{\mathrm{i}j(\Omega+\omega)\Delta t}\sum_{l=0}^{N-1}\mathrm{e}^{\mathrm{i}l(-\omega'-\Omega)\Delta t}$$

$$+ \sum_{j=0}^{N-1}\mathrm{e}^{\mathrm{i}j(\Omega-\omega)\Delta t}\sum_{l=0}^{N-1}\mathrm{e}^{\mathrm{i}l(\omega'-\Omega)\Delta t}$$

$$\left.- \sum_{j=0}^{N-1}\mathrm{e}^{\mathrm{i}j(\Omega-\omega)\Delta t}\sum_{l=0}^{N-1}\mathrm{e}^{\mathrm{i}l(-\omega'-\Omega)\Delta t}\right]S_{\alpha,\beta}(\Omega)\,\mathrm{d}\Omega$$

$$= -\mathrm{i}\int_{-\infty}^{\infty}H^N_{\alpha,\beta}(\Omega;\omega,\omega')\,\mathrm{d}\Omega$$

$$+ \mathrm{i}\int_{-\infty}^{\infty}H^N_{\alpha,\beta}(\Omega;\omega,-\omega')\,\mathrm{d}\Omega$$

$$- \mathrm{i}\int_{-\infty}^{\infty}H^N_{\alpha,\beta}(\Omega;-\omega,\omega')\,\mathrm{d}\Omega$$

$$+ \mathrm{i}\int_{-\infty}^{\infty}H^N_{\alpha,\beta}(\Omega;-\omega,-\omega')\,\mathrm{d}\Omega \qquad (28)$$

where $H^N_{\alpha,\beta}$ is given by Eq. (17).

Again, $\int_{-\infty}^{\infty}H^N_{\alpha,\beta}(\Omega;\omega,\omega')\,\mathrm{d}\Omega$, $\int_{-\infty}^{\infty}H^N_{\alpha,\beta}(\Omega;\omega,-\omega')\,\mathrm{d}\Omega$, $\int_{-\infty}^{\infty}H^N_{\alpha,\beta}(\Omega;-\omega,\omega')\,\mathrm{d}\Omega$ and

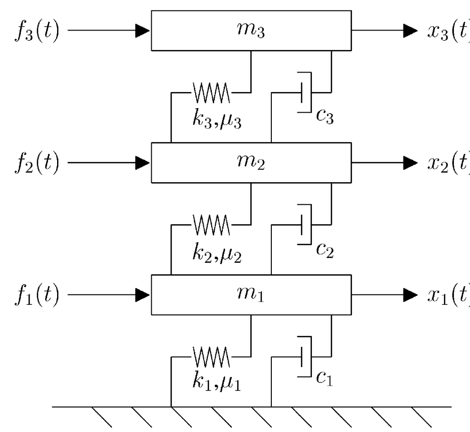

Fig. 1. Three-DOF Duffing oscillator.

$\int_{-\infty}^{\infty}H^N_{\alpha,\beta}(\Omega;-\omega,-\omega')\,\mathrm{d}\Omega$ tend to zero as $N\to\infty$, so

$$\lim_{N\to\infty}E\left[\left(R^N_\alpha(\omega) - E\left[R^N_\alpha(\omega)\right]\right)\left(I^N_\beta(\omega') - E\left[I^N_\beta(\omega')\right]\right)\right]$$

$$= 0, \qquad \text{if } \omega \neq \omega' \qquad (29)$$

4.4. $S^N_{\alpha,\beta}(\omega)$ with $S^N_{\gamma,\delta}(\omega')$

We conclude that any element in the set $\{R^N_\alpha(\omega), I^N_\alpha(\omega), R^N_\beta(\omega), I^N_\beta(\omega)\}$ with any element in the set $\{R^N_\gamma(\omega'), I^N_\gamma(\omega'), R^N_\delta(\omega'), I^N_\delta(\omega')\}$ gives an uncorrelated pair, where $\alpha,\beta,\gamma,\delta = 1,2,\ldots,d$ and $\omega \neq \omega'$. Furthermore, $R^N_\alpha(\Omega)$ and $I^N_\alpha(\Omega)$ are Gaussian distributed $\forall\Omega\in\mathbb{R}$ and $\alpha = 1,2,\ldots,d$ as $N\to\infty$ even if the stochastic process $\mathbf{x}$ is not Gaussian (Section 3). Since uncorrelated Gaussian random variables are independent, as $N\to\infty$ each element in the set $\{R^N_\alpha(\omega), I^N_\alpha(\omega), R^N_\beta(\omega), I^N_\beta(\omega)\}$ is statistically independent of each element in the set $\{R^N_\gamma(\omega'), I^N_\gamma(\omega'), R^N_\delta(\omega'), I^N_\delta(\omega')\}$, where $\alpha,\beta,\gamma,\delta = 1,2,\ldots,d$ and $\omega \neq \omega'$. By using Eqs. (1)–(3), the following can be obtained:

$$S^N_{\alpha,\beta}(\omega) = \frac{\Delta t}{2\pi}X^N_\alpha(\omega)X^{N*}_\beta(\omega)$$

$$= \frac{\Delta t}{2\pi}\left\{\left[R^N_\alpha(\omega)R^N_\beta(\omega) + I^N_\alpha(\omega)I^N_\beta(\omega)\right]\right. \qquad (30)$$

$$\left.+\mathrm{i}\left[I^N_\alpha(\omega)R^N_\beta(\omega) - R^N_\alpha(\omega)I^N_\beta(\omega)\right]\right\}$$

Therefore, $S^N_{\alpha,\beta}(\omega)$ and $S^N_{\gamma,\delta}(\omega')$ are statistically independent if $\alpha,\beta,\gamma,\delta = 1,2,\ldots,d$, $\omega \neq \omega'$ and $0 < \omega, \omega' < \pi/\Delta t$, the Nyquist frequency.

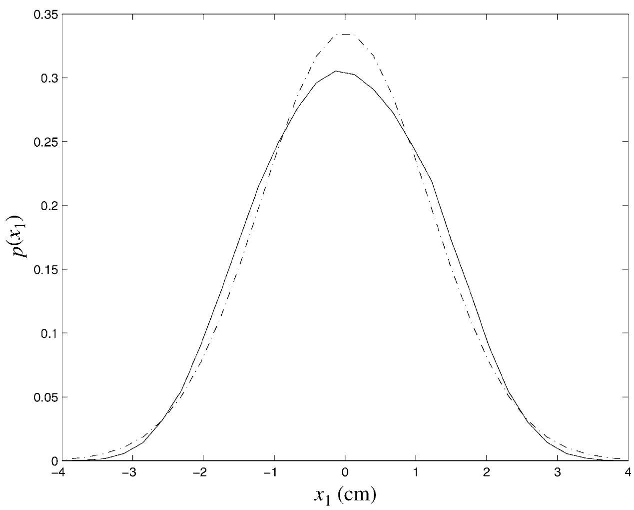

Fig. 2. PDF of the system response at the 1st DOF.

5. Illustrative example

Consider a three-degree-of-freedom Duffing oscillator as shown in Fig. 1:

$$\mathbf{M}\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{bmatrix}+\begin{bmatrix}c_1+c_2 & -c_2 & 0\\ -c_2 & c_2+c_3 & -c_3\\ 0 & -c_3 & c_3\end{bmatrix}\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}$$

$$+\begin{pmatrix}k_1+k_2 & -k_2 & 0\\ -k_2 & k_2+k_3 & -k_3\\ 0 & -k_3 & k_3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$$

$$+\begin{bmatrix}\mu_1 x_1^3+\mu_2(x_1-x_2)^3\\ \mu_2(x_2-x_1)^3+\mu_3(x_2-x_3)^3\\ \mu_3(x_3-x_2)^3\end{bmatrix}$$

$$=\begin{bmatrix}f_1(t)\\ f_2(t)\\ f_3(t)\end{bmatrix} \tag{31}$$

where $\mathbf{M}=\mathbf{I}$ kg is the mass matrix, $c_j = 2.0\times10^{-3}$ N s/cm, $k_j = 0.5$ N/cm and $\mu_j = 0.1$ N/cm^3 are the damping coefficient, linear stiffness and third order stiffness for the jth floor

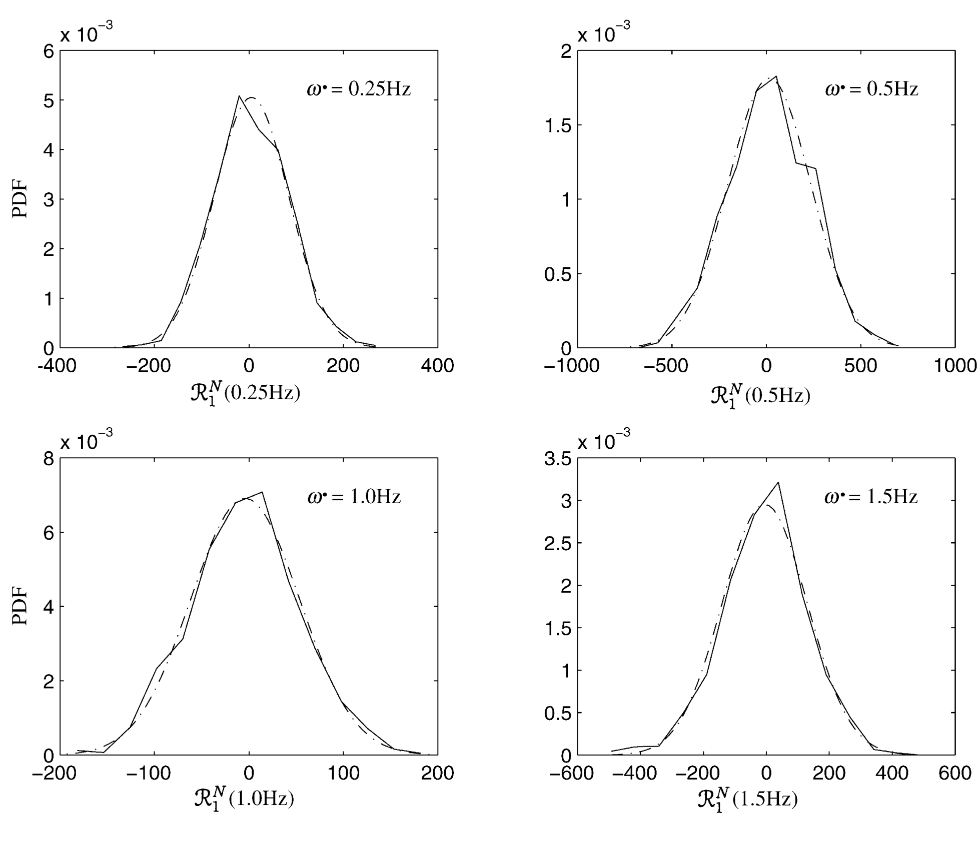

Fig. 3. PDF of the real part of the FFT for the 1st DOF response.

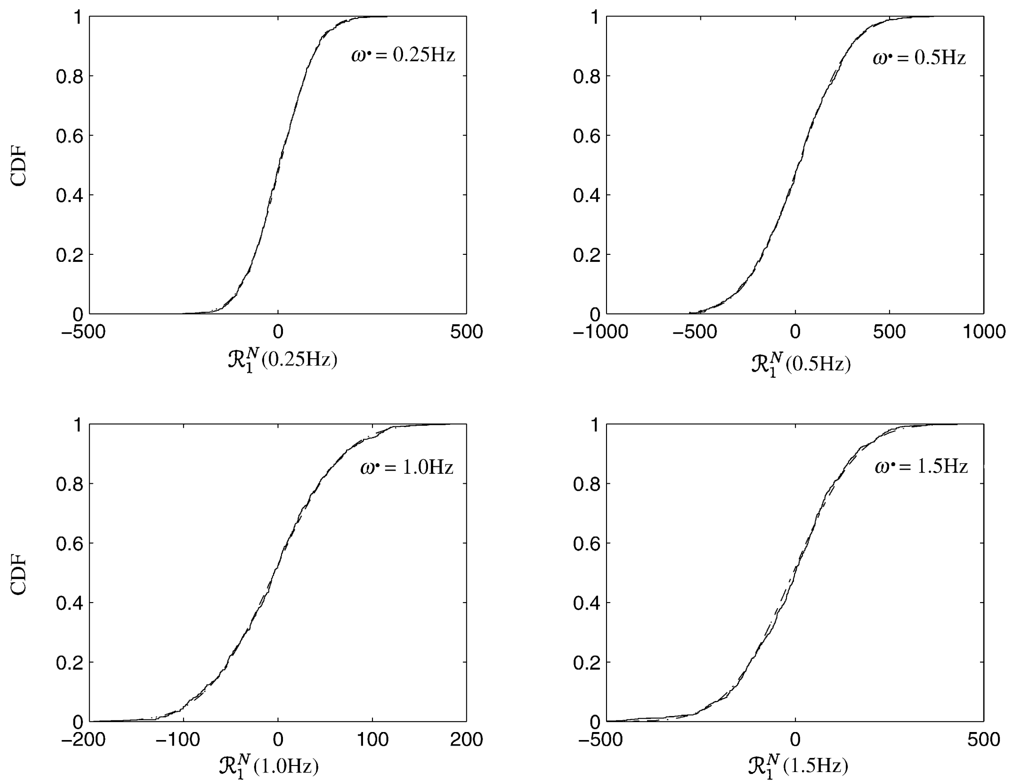

Fig. 4. CDF of the real part of the FFT for the 1st DOF response.

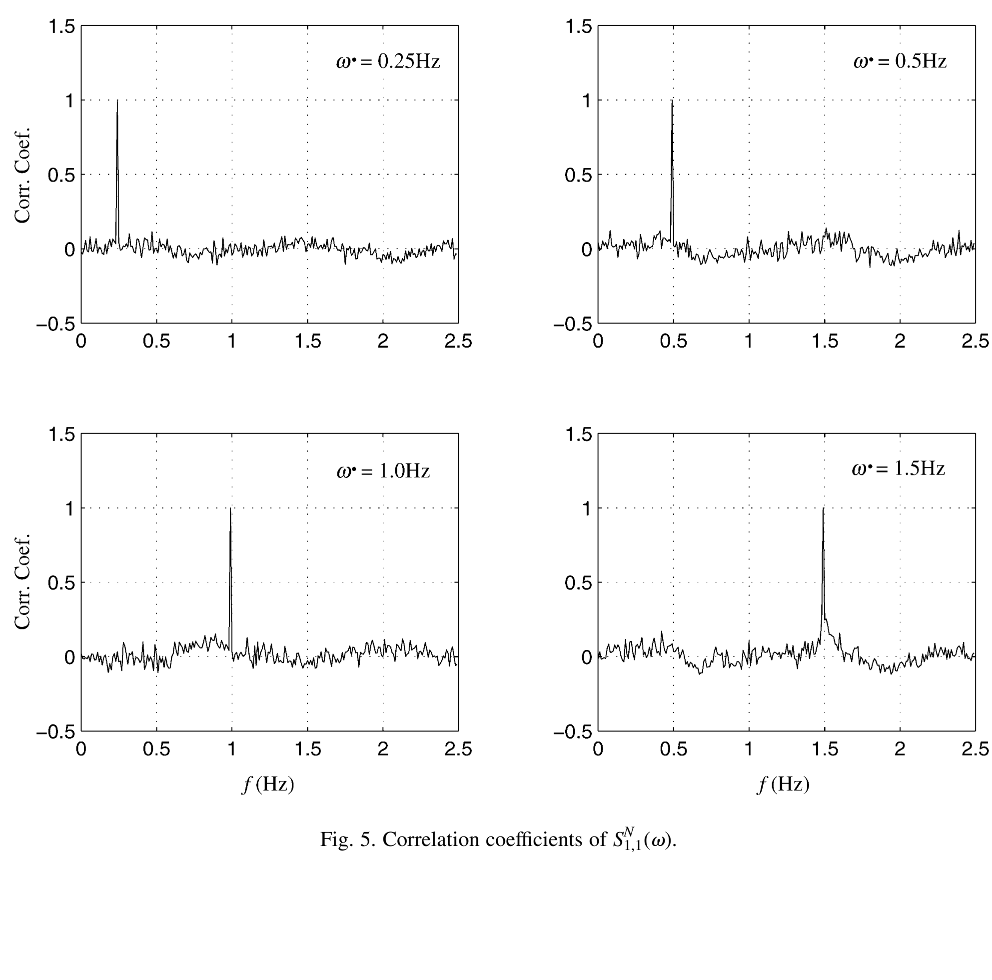

Fig. 5. Correlation coefficients of $S_{1,1}^N(\omega)$.

($j = 1, 2, 3$), respectively. The small amplitude fundamental frequency of the structure is 0.5 Hz. The external force $[f_1, f_2, f_3]^T$ is assumed to be independent Gaussian white noise with spectral intensity 5.0×10^{-4} N^2/s.

The response time history is simulated using the function 'ODE45' in Matlab [3]. The duration and sampling time interval are 5000 s and 0.02 s, respectively. Fig. 2 shows the PDF of the 1st DOF response $p(x_1)$, estimated using 250 000 samples (solid) and its best fitting curve in the class of Gaussian distributions. It can be seen that the structural response is non-Gaussian [9]. The standard deviation of these samples is 1.2 cm.

Two thousand realizations, each with duration 100 s and sampling time interval 0.02 s, are generated and their FFTs are calculated. Fig. 3 shows the PDF for $R_1^N(\omega)$ at $\omega = 0.25$, 0.5, 1.0, 1.5 Hz. It can be seen that the Gaussian approximation for the real part of the FFT is accurate although the structural response is not Gaussian. Furthermore, the corresponding cumulative distribution function (CDF) is shown in Fig. 4. The solid line corresponds to the best fitting Gaussian CDF and the dashed line corresponds to the CDF estimated using simulation. It can be seen that the two curves lie virtually on top of each other, implying that the real part of the FFT of the spectral density is well approximated by the Gaussian distribution. The same conclusion can be drawn for the imaginary part and for the response of other floors.

Fig. 5 shows the sample correlation coefficients for the spectral density of the response at the 1st DOF $S_{1,1}^N(\omega)$ at 0.25, 0.5, 1.0, 1.5 Hz with other frequencies. It can be seen that the coefficients of correlation are around zero at each frequency, except when the frequencies match where it is unity. The same conclusion can be drawn for the other components (auto or cross-terms) of the spectral density matrix.

6. Concluding remarks

A spectral density matrix estimator is defined based on a finite number, N, of data points which takes care of aliasing and leakage effects automatically. Furthermore, the probability density function of this spectral density matrix estimator for a general stationary stochastic vector process as $N \to \infty$ is presented. It is also proved that for such a process, the spectral density estimators corresponding to different frequencies are asymptotically independent as $N \to \infty$. This implies that the spectral density estimators of the response of a linear or non-linear dynamical system subject to input modeled as a stationary stochastic process are always asymptotically independent at different frequencies as $N \to \infty$.

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