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Fourth order finite difference schemes for time–space fractional sub-diffusion equations

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ABSTRACT

In this paper, we devote to the study of high order finite difference schemes for one- and two-dimensional time–space fractional sub-diffusion equations. A fourth order finite difference scheme is invoked for the spatial fractional derivatives, and the $L1$ approximation is applied to the temporal fractional parts. For the two-dimensional case, an alternating direction implicit scheme based on $L1$ approximation is proposed. The stability and convergence of the proposed methods are studied. Numerical experiments are performed to verify the effectiveness and accuracy of the proposed difference schemes.

1. Introduction

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders. During the past decades, fractional calculus has gained great popularity due to its widespread applications in fields of science and engineering [1–5]. One of its most important applications is to describe the sub-diffusion and super-diffusion process [6–9]. The suitable mathematical models are the time and/or space fractional diffusion equations, where the classical first order derivative in time is replaced by the Caputo fractional derivative [10] of order $\gamma \in (0, 1)$, and the second order derivative in space is essentially replaced by the Riemann–Liouville fractional derivative [10] of order $\alpha \in (1, 2)$. It is well known that the analytical solutions to the fractional differential equations are usually difficult to derive and always contain some infinite series even if it is luckily obtained, which make evaluation very expensive. Therefore, the development of numerical methods for these problems has received enormous attention and undergone a fast evolution in recent years [11,6,12–16,9,17–24].

Among a variety of techniques developed for fractional differential equations, the finite difference method should be the most popular one because it is direct and convenient to use. Meerschaert and Tadjeran [18] initially proposed a shifted Grünwald–Letnikov discretization to approximate the space fractional differential equations with a left sided Riemann–Liouville fractional derivative, which they showed to be stable and first-order accuracy in space. Extensions of this scheme to address various space fractional differential equations [25,19,23,26] followed soon after. Recently, Deng and his co-workers [27–29] exploited the weighted and shifted skill to construct a series of high-order finite difference schemes,

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named weighted and shifted Grünwald difference (WSGD) approximations, to the Riemann–Liouville space fractional derivatives. Motivated by this idea, Hao, Sun, and Cao [30] proposed a fourth-order quasi-compact difference scheme by carefully weighting the Grünwald approximation formula with different shifts and combining the compact technique for solving space fractional differential equations. For time fractional differential equations, the $L1$ formula [31] should be the major numerical differentiation formula to directly discretize the temporal fractional derivatives. Based on the $L1$ approximation, many stable numerical schemes have been established and analyzed [32–34,13,35,36] in the past decade.

Besides time or space fractional differential equations, the fractional differential equations with both temporal and spatial fractional derivatives have also received an increasing attention in recent years and have been used to model a wide range of phenomenons [37–42,21,22,43]. Therefore, the design of efficient and stable numerical schemes for time–space fractional differential equations is also an important activity. Liu et al. [44] proposed an implicit finite difference approximation to the time–space fractional diffusion equation, where the unconditional stability and first-order accuracy in both time and space were proved. Yang et al. [45] derived a novel numerical method based on the matrix transfer technique in space and finite difference scheme (or Laplace transform) in time to deal with the time–space fractional diffusion equations in two dimensions. Chen, Deng, and Wu [6] applied the $L1$ approximation to the time fractional derivative and second-order finite difference discretizations to the space fractional derivative for solving the two-dimensional time–space Caputo–Riesz fractional diffusion equation with variable coefficients in a finite domain. Most recently, an alternating direction implicit (ADI) scheme with second-order accuracy in both time and space is constructed to the time–space Caputo–Riesz fractional diffusion-wave equation by Wang, Vong, and Lei [46]. In the paper they also considered the time–space fractional sub-diffusion equation and constructed a full discretization difference scheme without dimensional (directional) splitting by the $L1$ formulae in time and second-order approximation in space.

In this paper, we focus on the high-order finite difference schemes for time–space fractional sub-diffusion equations. The proposed schemes are based on using the fourth-order quasi-compact difference scheme proposed by Hao, Sun, and Cao [30] for spatial approximation, which needs fewer grid points to produce a high accuracy solution. For the temporal discretization, we adopt the $L1$ approximation. Both one- and two-dimensional time–space fractional diffusion equations are considered. For the two-dimensional case, we also construct an ADI scheme based on the $L1$ approximation to reduce the storage requirement and the computational burden. Theoretical analyses show that the proposed schemes for both one- and two-dimensional cases are unconditionally stable and convergent.

The paper is organized as follows. In Section 2, we introduce the approximation of fourth-order quasi-compact finite difference scheme for Riemann–Liouville fractional derivatives and the $L1$ approximation to Caputo fractional derivatives. In Section 3, we apply these approximations to construct a full discretization scheme for the one-dimensional time–space fractional diffusion equation. The stability and convergence of the proposed scheme are discussed. In Section 4, we extend the discretization scheme to two-dimensional case. An ADI scheme based on $L1$ approximation is derived and the stability and convergence of the scheme are rigorously proved. Numerical examples are presented in Section 5 to support our theoretical analysis. Finally, concluding remarks are offered in Section 6.

2. Finite difference approximations of spatial and temporal fractional derivatives

We first introduce some definitions of fractional derivatives and then present their finite difference approximations.

Definition 2.1 (*[10]*)**.** For $\alpha \in (n-1, n)$ $(n \in \mathbb{N}^+)$, let $u(x)$ be $(n-1)$-times continuously differentiable on (a, ∞) (or $(-\infty, b)$ corresponding to the right derivative) and its n-times derivative be integrable on any subinterval of $[a, \infty)$ (or $(-\infty, b]$ corresponding to the right derivative). Then the left and right Riemann–Liouville fractional derivatives of the function $u(x)$ are defined as

$$ {}_a\mathcal{D}_x^{\alpha} u(x) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x \frac{u(\xi)}{(x-\xi)^{\alpha-n+1}} \mathrm{d}\xi $$

and

$$ {}_x\mathcal{D}_b^{\alpha} u(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^b \frac{u(\xi)}{(\xi-x)^{\alpha-n+1}} \mathrm{d}\xi, $$

respectively.

We remark that the 'a' in the definition can be extended to be '$-\infty$' and 'b' to be '$+\infty$'. In the following discussion, we assume that $u(x)$ is defined on $[a, b]$ and whenever necessary $u(x)$ can be smoothly zero extended to $(-\infty, b)$ or $(a, +\infty)$ or even $(-\infty, +\infty)$.

Definition 2.2 (*[10]*)**.** For $\gamma \in (n-1, n)$ $(n \in \mathbb{N}^+)$, let $u(t)$ be $(n-1)$-times continuously differentiable on $(0, \infty)$ and its n-times derivative be integrable on any subinterval of $[0, \infty)$. Then the Caputo fractional derivative of the function $u(t)$ is defined as

$$ {}_0^C\mathcal{D}_t^{\gamma} u(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{u^{(n)}(\zeta)}{(t-\zeta)^{\gamma-n+1}} \mathrm{d}\zeta, \quad t \in (0, \infty). $$

Most recently, Hao, Sun, and Cao [30] established a fourth-order quasi-compact finite difference scheme to Riemann–Liouville fractional derivatives. The essential idea is to vanish the low order leading terms in asymptotic expansions for the truncation errors by the weighted average. Let the WSGD operators for $1 < \alpha \le 2$ be defined by

$$\delta^{\alpha}_{x,\pm}u(x) = \lambda_1 A^{\alpha}_{\pm,h,1}u(x) + \lambda_0 A^{\alpha}_{\pm,h,0}u(x) + \lambda_{-1}A^{\alpha}_{\pm,h,-1}u(x), \tag{2.1}$$

where

$$h > 0, \qquad \lambda_1 = \frac{\alpha^2 + 3\alpha + 2}{12}, \qquad \lambda_0 = \frac{4 - \alpha^2}{6}, \qquad \lambda_{-1} = \frac{\alpha^2 - 3\alpha + 2}{12},$$

and

$$A^{\alpha}_{\pm,h,r}u(x) = \frac{1}{h^{\alpha}}\sum_{k=0}^{\infty} g^{(\alpha)}_k u(x \mp (k-r)h) \quad \text{for } r = 1, 0, -1, \tag{2.2}$$

are the first order shift Grünwald difference operator to Riemann–Liouville fractional derivatives [18] with coefficients $g^{(\alpha)}_0 = 1$, and

$$g^{(\alpha)}_k = \frac{(-1)^k}{k!}\alpha(\alpha-1)\cdots(\alpha-k+1), \quad k = 1, 2, \ldots.$$

In [30], it was showed that the operators in (2.1) have second order accuracy for approximating Riemann–Liouville fractional derivatives. Letting the second order central difference operator $\delta^2_x u(x) = [u(x-h) - 2u(x) + u(x+h)]/h^2$ and denoting the finite difference operator

$$\mathcal{H}_{\alpha}u(x) = \left(1 + c_{\alpha}h^2\delta^2_x\right)u(x) \quad \text{with } c_{\alpha} = \frac{-\alpha^2 + \alpha + 4}{24},$$

they derived the following fourth-order approximations to Riemann–Liouville fractional derivatives.

Lemma 2.1 ([30]). *Let $u(x) \in L_1(\mathbb{R})$ and $u(x) \in \mathfrak{L}^{4+\alpha}(\mathbb{R})$. Then for a fixed h, we have*

$$\mathcal{H}_{\alpha}\left({}_{-\infty}\mathcal{D}^{\alpha}_x u(x)\right) = \delta^{\alpha}_{x,+}u(x) + \mathcal{O}(h^4),$$
$$\mathcal{H}_{\alpha}\left({}_{x}\mathcal{D}^{\alpha}_{+\infty} u(x)\right) = \delta^{\alpha}_{x,-}u(x) + \mathcal{O}(h^4).$$

The symbol $\mathfrak{L}^{4+\alpha}(\mathbb{R})$ in the above lemma refers to

$$\mathfrak{L}^{4+\alpha}(\mathbb{R}) = \left\{ u \,\middle|\, \int_{-\infty}^{+\infty}(1+|\tau|)^{4+\alpha}|\hat{u}(\tau)|\mathrm{d}\tau < \infty \right\},$$

where $\hat{u}(\tau) = \int_{-\infty}^{+\infty} e^{\mathrm{i}\tau x}u(x)\mathrm{d}x$ is the Fourier transformation of $u(x)$.

Using (2.2), the WSGD operators $\delta^{\alpha}_{x,\pm}u(x)$ in (2.1) can be rewritten as

$$\delta^{\alpha}_{x,\pm}u(x) = \frac{1}{h^{\alpha}}\sum_{k=0}^{\infty} w^{(\alpha)}_k u(x \mp (k-1)h), \tag{2.3}$$

where

$$w^{(\alpha)}_0 = \lambda_1 g^{(\alpha)}_0, \qquad w^{(\alpha)}_1 = \lambda_1 g^{(\alpha)}_1 + \lambda_0 g^{(\alpha)}_0,$$
$$w^{(\alpha)}_k = \lambda_1 g^{(\alpha)}_k + \lambda_0 g^{(\alpha)}_{k-1} + \lambda_{-1}g^{(\alpha)}_{k-2}, \quad k \ge 2.$$

Following [30], for $u(x) \in \mathbb{C}[a, b]$ with $u(a) = u(b) = 0$, we make zero-extension such that $u(x)$ is defined on $\mathbb{R}$. Suppose $u(x) \in \mathfrak{L}^{4+\alpha}(\mathbb{R})$, one can get by Lemma 2.1 that

$$\mathcal{H}_{\alpha}({}_{a}\mathcal{D}^{\alpha}_x u(x)) = \frac{1}{h^{\alpha}}\sum_{k=0}^{\left\lfloor \frac{x-a}{h}\right\rfloor} w^{(\alpha)}_k u(x-(k-1)h) + \mathcal{O}(h^4),$$

$$\mathcal{H}_{\alpha}({}_{x}\mathcal{D}^{\alpha}_b u(x)) = \frac{1}{h^{\alpha}}\sum_{k=0}^{\left\lfloor \frac{b-x}{h}\right\rfloor} w^{(\alpha)}_k u(x+(k-1)h) + \mathcal{O}(h^4).$$

For the Caputo fractional derivative, there exists the following $L1$ approximation.

Lemma 2.2 ([47]). *Suppose $0 < \gamma < 1$, $u(t) \in \mathbb{C}^2[0, T]$. Take an integer n, let $n = t/\tau$ ($\tau > 0$), and denote*

$$\mathcal{D}_t^{\gamma} u(t) = \frac{\tau^{-\gamma}}{\Gamma(2-\gamma)} \left[u(t) - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) u(k\tau) - a_{n-1} u(0) \right] \tag{2.4}$$

where $a_k = (k+1)^{1-\gamma} - k^{1-\gamma}$. Then

$$\left| {}_0^C\mathcal{D}_t^{\gamma} u(t) - \mathcal{D}_t^{\gamma} u(t) \right| \le \frac{1}{\Gamma(2-\gamma)} \left[\frac{1-\gamma}{12} + \frac{2^{2-\gamma}}{2-\gamma} - (1 + 2^{-\gamma}) \right] \max_{0 \le s \le t} |u''(s)| \tau^{2-\gamma}.$$

In the following, we apply the above fourth-order finite difference approximation and the $L1$ approximation to construct full discretization schemes for time–space fractional sub-diffusion equations.

3. One-dimensional time–space fractional diffusion equation

In this section, we consider the following one-dimensional time–space fractional diffusion equation

$${}_0^C\mathcal{D}_t^{\gamma} u(x,t) = K_1^{+}\, {}_a\mathcal{D}_x^{\alpha} u(x,t) + K_1^{-}\, {}_x\mathcal{D}_b^{\alpha} u(x,t) + f(x,t), \quad (x,t) \in (a,b) \times (0,T], \tag{3.1}$$

$$u(a,t) = \phi_1(t), \qquad u(b,t) = \phi_2(t), \quad t \in (0,T], \tag{3.2}$$

$$u(x,0) = u_0(x), \quad x \in [a,b], \tag{3.3}$$

where ${}_0^C\mathcal{D}_t^{\gamma}$ is the Caputo fractional derivative with $0 < \gamma < 1$, ${}_a\mathcal{D}_x^{\alpha}$ and ${}_x\mathcal{D}_b^{\alpha}$ are the left and right Riemann–Liouville fractional derivatives with $1 < \alpha \le 2$, respectively. The diffusion coefficients K_1^{+} and K_1^{-} are nonnegative constants with $K_1^{+} + K_1^{-} \ne 0$. If $K_1^{+} \ne 0$, then $\phi_1(t) \equiv 0$, and if $K_1^{-} \ne 0$, then $\phi_2(t) \equiv 0$. We remark that if $K_1^{+} = K_1^{-}$, then the above equation reduces to the one-dimensional time–space Caputo–Riesz fractional diffusion equation [6]. Assume (3.1)–(3.3) have a unique solution $u \in \mathbb{C}_{x,t}^{6,2}([a,b] \times [0,T])$. Define

$$\hat{u}(x,t) = \begin{cases} u(x,t), & (x,t) \in [a,b] \times [0,T], \\ 0, & \text{others}. \end{cases}$$

Suppose for any fixed $t \in (0,T]$, $\hat{u}(x,t) \in \mathfrak{L}^{4+\alpha}(\mathbb{R})$. Throughout this paper, we assume that these conditions are satisfied when referring to the solution of (3.1)–(3.3).

3.1. Derivation of the finite difference scheme

Define the uniform time stepsize as $\tau = T/N$ with N being a positive integer and let $t_n = n\tau$, $0 \le n \le N$. The time domain $[0,T]$ is covered by $\Omega_\tau = \{t_n | 0 \le n \le N\}$. Given grid function $w = \{w^n | 0 \le n \le N\}$ on Ω_τ.

For spatial approximation, let $h = (b-a)/M$ with a positive integer M and take the mesh points $x_i = a + ih$, $0 \le i \le M$. The spatial domain $[a,b]$ is covered by $\Omega_h = \{x_i | 0 \le i \le M\}$. For any grid function $v = \{v_i | 0 \le i \le M\}$ on Ω_h, denote

$$\delta_x^2 v_i = \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}$$

and

$$\mathcal{H}_\alpha v_i = \begin{cases} (1 + c_\alpha h^2 \delta_x^2) v_i, & 1 \le i \le M-1, \\ v_i, & i = 0 \text{ or } M. \end{cases}$$

For convenience of presentation, let

$$\mathcal{D}_x^{\alpha} = (K_1^{+}\, {}_a\mathcal{D}_x^{\alpha} + K_1^{-}\, {}_x\mathcal{D}_b^{\alpha}), \qquad \delta_x^{\alpha} = (K_1^{+} \delta_{x,+}^{\alpha} + K_1^{-} \delta_{x,-}^{\alpha}). \tag{3.4}$$

Considering (3.1) at the point (x_i, t_n), we have

$${}_0^C\mathcal{D}_t^{\gamma} u(x_i, t_n) = \mathcal{D}_x^{\alpha} u(x_i, t_n) + f(x_i, t_n), \quad 0 \le i \le M, \; 1 \le n \le N.$$

Acting the operator $\mathcal{H}_\alpha$ on both sides of the above equation gives

$$\mathcal{H}_\alpha({}_0^C\mathcal{D}_t^{\gamma} u(x_i, t_n)) = \mathcal{H}_\alpha(\mathcal{D}_x^{\alpha} u(x_i, t_n)) + \mathcal{H}_\alpha f(x_i, t_n), \quad 1 \le i \le M-1, \; 1 \le n \le N. \tag{3.5}$$

Define the grid functions

$$U_i^n = u(x_i, t_n), \quad f_i^n = f(x_i, t_n), \; 0 \le i \le M, \; 0 \le n \le N.$$

Then by applying Lemmas 2.1 and 2.2 to (3.5), we obtain that

$$\frac{\tau^{-\gamma}}{\Gamma(2-\gamma)}\mathcal{H}_\alpha\left[U_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})U_i^k-a_{n-1}U_i^0\right]=\delta_x^\alpha U_i^n+\mathcal{H}_\alpha f_i^n+R_i^n, \tag{3.6}$$

where there exists a constant c_1 such that

$$|R_i^n|\le c_1(\tau^{2-\gamma}+h^4),\quad 1\le i\le M-1,\ 1\le n\le N. \tag{3.7}$$

Let $\mu=\tau^\gamma\Gamma(2-\gamma)$. Omitting the small terms R_i^n in (3.6) and denoting by u_i^n the numerical approximation of U_i^n, we can construct the finite difference scheme for solving Eq. (3.1) with initial and boundary conditions of (3.3) and (3.2) as follows:

$$\mathcal{H}_\alpha\left[u_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})u_i^k-a_{n-1}u_i^0\right]=\mu\delta_x^\alpha u_i^n+\mu\mathcal{H}_\alpha f_i^n,\quad 1\le i\le M-1,\ 1\le n\le N, \tag{3.8}$$

$$u_0^n=\phi_1(t_n),\qquad u_M^n=\phi_2(t_n),\quad 1\le n\le N, \tag{3.9}$$

$$u_i^0=u_0(x_i),\quad 0\le i\le M. \tag{3.10}$$

3.2. Stability and convergence analysis of the difference scheme

Next, we analyze the stability and convergence for the scheme (3.8)–(3.10). Let

$$\mathcal{V}_h=\{v|v=(v_0,v_1,\ldots,v_M),\ v_0=v_M=0\}$$

be space grid functions defined on Ω_h. For any $u,v\in\mathcal{V}_h$, we define

$$(u,v)=h\sum_{i=1}^{M-1}u_iv_i$$

and corresponding discrete L^2 norm

$$\|v\|=\sqrt{(v,v)}.$$

Some lemmas are needed for analyzing the stability and convergence of the finite difference scheme.

Lemma 3.3 ([12,35]). *Let $0<\gamma<1$, $a_k=(k+1)^{1-\gamma}-k^{1-\gamma}$, $k=0,1,\ldots$. Then*
(1) $1=a_0>a_1>a_2>\cdots>a_n>\cdots\to 0$;
(2) $(1-\gamma)(k+1)^{-\gamma}<a_k<(1-\gamma)k^{-\gamma}$;
(3) $\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})+a_{n-1}=1$.

Lemma 3.4 ([30]). *For any $u,v\in\mathcal{V}_h$, it holds that*
(1) $(\delta_{x,+}^\alpha u,v)=(u,\delta_{x,-}^\alpha v)$;
(2) $(\delta_{x,+}^\alpha v,v)\le 0$, $(\delta_{x,-}^\alpha v,v)\le 0$.

Lemma 3.5 ([30]). *For any $u,v\in\mathcal{V}_h$, it holds that*
(1) $(\mathcal{H}_\alpha u,v)=(u,\mathcal{H}_\alpha v)$;
(2) $\frac{1}{3}\|v\|^2\le(\mathcal{H}_\alpha v,v)\le\|v\|^2$.

From Lemma 3.5 we know that $\mathcal{H}_\alpha$ is positive definite and self-adjoint. Therefore we can consider its square root denoted as Q_α. Obviously Q_α is also a positive definite and self-adjoint operator, then we can define its inverse operator as Q_α^{-1}.

Lemma 3.6. *For any $v\in\mathcal{V}_h$, it holds that*

$$\frac{1}{\sqrt{3}}\|v\|\le\|Q_\alpha v\|\le\|v\|,\qquad \|v\|\le\|Q_\alpha^{-1}v\|\le\sqrt{3}\|v\|. \tag{3.11}$$

Proof. It is clear that

$$(\mathcal{H}_\alpha v,v)=(Q_\alpha^2v,v)=(Q_\alpha v,Q_\alpha v)=\|Q_\alpha v\|^2. \tag{3.12}$$

Then by (2) of Lemma 3.5, we have

$$\frac{1}{\sqrt{3}}\|v\|\le\|Q_\alpha v\|\le\|v\|.$$

Replacing v in the above inequality by $Q_\alpha^{-1}v$, we obtain the second inequality of (3.11). □

Now we give a prior estimate for the finite difference scheme (3.8)–(3.10).

Lemma 3.7. *Suppose* $\{v_i^n\}$ *be the solution of*

$$\mathcal{H}_\alpha\left[v_i^n-\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})v_i^k-a_{n-1}v_i^0\right]=\mu\delta_x^\alpha v_i^n+\mu p_i^n,\quad 1\le i\le M-1,\ 1\le n\le N, \tag{3.13}$$

$$v_0^n=0,\qquad v_M^n=0,\quad 1\le n\le N, \tag{3.14}$$

$$v_i^0=v_0(x_i),\quad 0\le i\le M, \tag{3.15}$$

then

$$\|v^n\|\le\sqrt{3}\left(\|v^0\|+\sqrt{3}\Gamma(1-\gamma)T^\gamma\max_{1\le l\le N}\|p^l\|\right),$$

where $v_0(x_0)=v_0(x_M)=0$, $\|p^l\|=\sqrt{h\sum_{i=1}^{M-1}(p_i^l)^2}$.

Proof. Taking the inner product of (3.13) with v^n, we have

$$(\mathcal{H}_\alpha v^n,v^n)-\mu(\delta_x^\alpha v^n,v^n)=\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(\mathcal{H}_\alpha v^k,v^n)+a_{n-1}(\mathcal{H}_\alpha v^0,v^n)+\mu(p^n,v^n). \tag{3.16}$$

By (3.12), we know that

$$(\mathcal{H}_\alpha v^n,v^n)=\|Q_\alpha v^n\|^2. \tag{3.17}$$

It follows from Lemma 3.4 and Eq. (3.4) that

$$-\mu(\delta_x^\alpha v^n,v^n)\ge 0. \tag{3.18}$$

As to the right hand side of (3.16), by the Cauchy–Schwarz inequality, we obtain

$$(\mathcal{H}_\alpha v^k,v^n)=(Q_\alpha v^k,Q_\alpha v^n)\le\|Q_\alpha v^k\|\cdot\|Q_\alpha v^n\|,\quad k=0,1,\dots, \tag{3.19}$$

and

$$(p^n,v^n)=(Q_\alpha^{-1}p^n,Q_\alpha v^n)\le\|Q_\alpha^{-1}p^n\|\cdot\|Q_\alpha v^n\|. \tag{3.20}$$

Substituting (3.17)–(3.20) into (3.16) and noticing that $(a_{n-k-1}-a_{n-k})$ and a_{n-1} are positive, we have

$$\|Q_\alpha v^n\|\le\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|Q_\alpha v^k\|+a_{n-1}\|Q_\alpha v^0\|+\mu\|Q_\alpha^{-1}p^n\|,\quad 1\le n\le N. \tag{3.21}$$

Since $\mu=\tau^\gamma\Gamma(2-\gamma)=\Gamma(1-\gamma)T^\gamma(1-\gamma)N^{-\gamma}$, according to (2) of Lemma 3.3, we have

$$\mu<\Gamma(1-\gamma)T^\gamma a_{n-1},\quad 1\le n\le N.$$

Denote

$$F=\|Q_\alpha v^0\|+\Gamma(1-\gamma)T^\gamma\max_{1\le l\le N}\|Q_\alpha^{-1}p^l\|.$$

Then by (3.21), we have

$$\|Q_\alpha v^n\|\le\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|Q_\alpha v^k\|+a_{n-1}F,\quad 1\le n\le N.$$

We use mathematical induction to prove that

$$\|Q_\alpha v^n\|\le F,\quad 1\le n\le N.$$

Obviously, it holds for $n=1$. Assume that

$$\|Q_\alpha v^k\|\le F,\quad 1\le k\le n-1.$$

By Lemma 3.3, we obtain

$$\|Q_\alpha v^n\|\le\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|Q_\alpha v^k\|+a_{n-1}F\le\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})F+a_{n-1}F=F.$$

Furthermore, it follows from Lemma 3.6 that

$$\|v^n\| \le \sqrt{3}\|Q_\alpha v^n\|, \qquad F \le \|v^0\| + \sqrt{3}\Gamma(1-\gamma)T^\gamma \max_{1\le l\le N} \|p^l\|,$$

which completes the proof. □

Using Lemma 3.7, we immediately have the following result.

Theorem 3.1. *The difference scheme (3.8)–(3.10) is unconditionally stable to the initial values u_0 and right hand side f for all $0 < \gamma < 1$ and $1 < \alpha \le 2$.*

Next, we consider the error estimate of (3.8)–(3.10). Let

$$e_i^n = U_i^n - u_i^n, \quad 0 \le i \le M,\ 0 \le n \le N.$$

The following theorem of convergence holds.

Theorem 3.2. *Assume that $\{U_i^n\}$ and $\{u_i^n\}$ are the exact solution of problem (3.1)–(3.3) and difference scheme (3.8)–(3.10), respectively. Then the following estimate*

$$\|e^n\| \le 3c_1\sqrt{b-a}\,\Gamma(1-\gamma)T^\gamma(\tau^{2-\gamma}+h^4), \quad 1 \le n \le N,$$

holds for all $0 < \gamma < 1$ and $1 < \alpha \le 2$.

Proof. By (3.6) and (3.8), the error equations can be obtained as follows:

$$\mathcal{H}_\alpha\left[e_i^n - \sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})e_i^k - a_{n-1}e_i^0\right] = \mu\delta_x^\alpha e_i^n + \mu R_i^n, \quad 1 \le i \le M-1,\ 1 \le n \le N,$$

$$e_0^n = 0, \qquad e_M^n = 0, \quad 1 \le n \le N,$$

$$e_i^0 = 0, \quad 0 \le i \le M.$$

It follows from Lemma 3.7 and the estimate (3.7) that

$$\|e^n\| \le 3\Gamma(1-\gamma)T^\gamma \max_{1\le l\le N}\|R^l\| \le 3c_1\sqrt{b-a}\,\Gamma(1-\gamma)T^\gamma(\tau^{2-\gamma}+h^4), \quad 1 \le n \le N. \quad \square$$

4. Two-dimensional time–space fractional diffusion equation

In this section, we consider the following two-dimensional time–space fractional diffusion equation

$$\begin{aligned} {}_0^C\mathcal{D}_t^\gamma u(x,y,t) &= K_1^+\,{}_a\mathcal{D}_x^\alpha u(x,y,t) + K_1^-\,{}_x\mathcal{D}_b^\alpha u(x,y,t) + K_2^+\,{}_c\mathcal{D}_y^\beta u(x,y,t) \\ &\quad + K_2^-\,{}_y\mathcal{D}_d^\beta u(x,y,t) + f(x,y,t), \quad (x,y,t) \in \Omega \times (0,T], \end{aligned} \tag{4.1}$$

$$u(x,y,t) = \phi(x,y,t), \quad (x,y,t) \in \partial\Omega \times (0,T], \tag{4.2}$$

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in \bar{\Omega}, \tag{4.3}$$

where ${}_0^C\mathcal{D}_t^\gamma$ is the Caputo fractional derivative with $0 < \gamma < 1$, ${}_a\mathcal{D}_x^\alpha$, ${}_x\mathcal{D}_b^\alpha$ and ${}_c\mathcal{D}_y^\beta$, ${}_y\mathcal{D}_d^\beta$ are the left and right Riemann–Liouville fractional derivatives with $1 < \alpha, \beta \le 2$ respectively, the domain $\Omega = (a,b) \times (c,d)$. The diffusion coefficients K_i^+ and K_i^- for $i = 1, 2$ are nonnegative constants with $K_i^+ + K_i^- \ne 0$. If $K_1^+ \ne 0$, then $\phi(a,y,t) \equiv 0$; if $K_1^- \ne 0$, then $\phi(b,y,t) \equiv 0$; if $K_2^+ \ne 0$, then $\phi(x,c,t) \equiv 0$; if $K_2^- \ne 0$, then $\phi(x,d,t) \equiv 0$. We note that if $K_1^+ = K_1^-$ and $K_2^+ = K_2^-$, then Eq. (4.1) is essentially the two-dimensional time–space Caputo–Riesz fractional diffusion equation with constant coefficients [6]. Similar to the one-dimensional case, we assume (4.1)–(4.3) have a unique solution $u(x,y,t) \in \mathbb{C}_{x,y,t}^{6,6,2}(\bar{\Omega}\times[0,T])$. For any $t \in [0,T]$, $y \in [c,d]$, define a function $\hat{u}(x)$ on $\mathbb{R}$ as follows:

$$\hat{u}(x) = \begin{cases} u(x,y,t), & x \in [a,b], \\ 0, & x \notin [a,b]. \end{cases}$$

For any $t \in [0,T]$, $x \in [a,b]$, define a function $\hat{v}(y)$ on $\mathbb{R}$ as follows:

$$\hat{v}(y) = \begin{cases} u(x,y,t), & y \in [c,d], \\ 0, & y \notin [c,d]. \end{cases}$$

For any $x \in [a,b]$, $y \in [c,d]$, define a function

$$\hat{w}(t) = u(x,y,t), \ t \in [0,T].$$

Assume that $\hat{u}(x)$, $\hat{v}(y)$ satisfy the conditions of Lemma 2.1 and $\hat{w}(t)$ satisfies the condition of Lemma 2.2.

4.1. Derivation of the finite difference scheme

For the spatial approximation, let $h_1 = (b-a)/M_1$, $h_2 = (d-c)/M_2$, and $h = \max\{h_1, h_2\}$ with positive integers M_1 and M_2. Take the mesh points $x_i = a + ih_1$, $i = 0, 1, 2, \ldots, M_1$ and $y_j = c + jh_2$, $j = 0, 1, 2, \ldots, M_2$. The spatial domain $\bar{\Omega}$ is covered by $\bar{\Omega}_h = \{(x_i, y_j) | 0 \le i \le M_1,\ 0 \le j \le M_2\}$. Let $\Omega_h = \bar{\Omega}_h \cap \Omega$ and $\partial\Omega_h = \bar{\Omega}_h \cap \partial\Omega$. For any grid function $v = \{v_{i,j} | 0 \le i \le M_1,\ 0 \le j \le M_2\}$, denote

$$\delta_x^2 v_{i,j} = \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h_1^2}, \qquad \delta_y^2 v_{i,j} = \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{h_2^2},$$

and

$$\mathcal{H}_1 v_{i,j} = \begin{cases} (1 + c_\alpha h_1^2 \delta_x^2) v_{i,j} & 1 \le i \le M_1 - 1,\ 0 \le j \le M_2, \\ v_{i,j} & i = 0 \text{ or } M_1,\ 0 \le j \le M_2, \end{cases}$$

$$\mathcal{H}_2 v_{i,j} = \begin{cases} (1 + c_\beta h_2^2 \delta_y^2) v_{i,j} & 1 \le j \le M_2 - 1,\ 0 \le i \le M_1, \\ v_{i,j} & j = 0 \text{ or } M_2,\ 0 \le i \le M_1. \end{cases}$$

Similar to Eq. (3.4), we introduce the notations

$$\mathcal{D}_y^\beta = K_2^+ {}_c\mathcal{D}_y^\beta + K_2^- {}_y\mathcal{D}_d^\beta, \qquad \delta_y^\beta = K_2^+ \delta_{y,+}^\beta + K_2^- \delta_{y,-}^\beta.$$

Considering (4.1) at the point (x_i, y_j, t_n), we have

$${}_0^C\mathcal{D}_t^\gamma u(x_i, y_j, t_n) = \mathcal{D}_x^\alpha u(x_i, y_j, t_n) + \mathcal{D}_y^\beta u(x_i, y_j, t_n) + f(x_i, y_j, t_n), \quad (x_i, y_j) \in \bar{\Omega}_h,\ 1 \le n \le N.$$

Performing the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ on both sides of the above equation yields

$$\begin{aligned} \mathcal{H}_1\mathcal{H}_2({}_0^C\mathcal{D}_t^\gamma u(x_i, y_j, t_n)) &= \mathcal{H}_1\mathcal{H}_2(\mathcal{D}_x^\alpha u(x_i, y_j, t_n)) + \mathcal{H}_1\mathcal{H}_2(\mathcal{D}_y^\beta u(x_i, y_j, t_n)) \\ &\quad + \mathcal{H}_1\mathcal{H}_2(f(x_i, y_j, t_n)), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N. \end{aligned} \tag{4.4}$$

Define the grid functions

$$U_{i,j}^n = u(x_i, y_j, t_n), \qquad f_{i,j}^n = f(x_i, y_j, t_n), \qquad (x_i, y_j) \in \bar{\Omega}_h,\ t_n \in \Omega_\tau.$$

By applying Lemmas 2.1 and 2.2 to (4.4), then we have

$$\mathcal{H}_1\mathcal{H}_2\mathcal{D}_t^\gamma U_{i,j}^n = \mathcal{H}_2\delta_x^\alpha U_{i,j}^n + \mathcal{H}_1\delta_y^\beta U_{i,j}^n + \mathcal{H}_1\mathcal{H}_2 f_{i,j}^n + R_{i,j}^n, \tag{4.5}$$

where there exists a constant c_2 such that

$$|R_{i,j}^n| \le c_2(\tau^{2-\gamma} + h_1^4 + h_2^4), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N.$$

Adding the small term $\mu^2\delta_x^\alpha\delta_y^\beta\mathcal{D}_t^\gamma U_{i,j}^n$ to Eq. (4.5), we get

$$(\mathcal{H}_1\mathcal{H}_2 + \mu^2\delta_x^\alpha\delta_y^\beta)\mathcal{D}_t^\gamma U_{i,j}^n = \mathcal{H}_2\delta_x^\alpha U_{i,j}^n + \mathcal{H}_1\delta_y^\beta U_{i,j}^n + \mathcal{H}_1\mathcal{H}_2 f_{i,j}^n + \hat{R}_{i,j}^n \tag{4.6}$$

with $\hat{R}_{i,j}^n = R_{i,j}^n + \mu^2\delta_x^\alpha\delta_y^\beta\mathcal{D}_t^\gamma U_{i,j}^n$. Noticing that $\mu = \tau^\gamma\Gamma(2-\gamma)$ and $0 < \gamma < 1$, we have

$$|\hat{R}_{i,j}^n| \le c_3(\tau^{\min\{2\gamma, 2-\gamma\}} + h_1^4 + h_2^4), \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \tag{4.7}$$

for a certain constant c_3. Omitting the small terms $\hat{R}_{i,j}^n$ in (4.6), denoting by $u_{i,j}^n$ the numerical approximation of $U_{i,j}^n$, and exploiting Eq. (2.4), we can construct the finite difference scheme for solving Eq. (4.1) with initial and boundary conditions of (4.3) and (4.2) as follows:

$$\begin{aligned} \mathcal{H}_1\mathcal{H}_2 u_{i,j}^n + \mu^2\delta_x^\alpha\delta_y^\beta u_{i,j}^n - \mu\mathcal{H}_2\delta_x^\alpha u_{i,j}^n - \mu\mathcal{H}_1\delta_y^\beta u_{i,j}^n &= \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(\mathcal{H}_1\mathcal{H}_2 + \mu^2\delta_x^\alpha\delta_y^\beta)u_{i,j}^k \\ &\quad + a_{n-1}(\mathcal{H}_1\mathcal{H}_2 + \mu^2\delta_x^\alpha\delta_y^\beta)u_{i,j}^0 + \mu\mathcal{H}_1\mathcal{H}_2 f_{i,j}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \end{aligned} \tag{4.8}$$

$$u_{i,j}^n = \phi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial\Omega_h,\ 1 \le n \le N, \tag{4.9}$$

$$u_{i,j}^0 = u_0(x_i, y_j), \quad (x_i, y_j) \in \bar{\Omega}_h. \tag{4.10}$$

Note that Eq. (4.8) can be factorized as

$$\begin{aligned} (\mathcal{H}_1 - \mu\delta_x^\alpha)(\mathcal{H}_2 - \mu\delta_y^\beta)u_{i,j}^n &= \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(\mathcal{H}_1\mathcal{H}_2 + \mu^2\delta_x^\alpha\delta_y^\beta)u_{i,j}^k \\ &\quad + a_{n-1}(\mathcal{H}_1\mathcal{H}_2 + \mu^2\delta_x^\alpha\delta_y^\beta)u_{i,j}^0 + \mu\mathcal{H}_1\mathcal{H}_2 f_{i,j}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N. \end{aligned}$$

Introducing an intermediate variable $\hat{u}_{i,j}^n$, we obtain the following ADI scheme:

$$(\mathcal{H}_1 - \mu\delta_x^\alpha)\hat{u}_{i,j}^n = \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(\mathcal{H}_1\mathcal{H}_2 + \mu^2\delta_x^\alpha\delta_y^\beta)u_{i,j}^k + a_{n-1}(\mathcal{H}_1\mathcal{H}_2 + \mu^2\delta_x^\alpha\delta_y^\beta)u_{i,j}^0 + \mu\mathcal{H}_1\mathcal{H}_2 f_{i,j}^n, \tag{4.11}$$

$$(\mathcal{H}_2 - \mu\delta_y^\beta)u_{i,j}^n = \hat{u}_{i,j}^n. \tag{4.12}$$

We remark that when $\hat{u}_{i,j}^n$ in (4.11) for $1 \le i \le M_1 - 1$ and fixed j are solved, the boundary values $\hat{u}_{0,j}^n$ and $\hat{u}_{M_1,j}^n$ are determined by (4.12) for $i = 0$ and $i = M_1$, respectively.

4.2. Stability and convergence analysis of the difference scheme

In the following, we give the stability and convergence analysis for the scheme (4.8)–(4.10). Similar to the one-dimensional case, let

$$\mathcal{V}_h = \{v \mid v = \{v_{i,j}\} \text{ is a grid function on } \bar{\Omega}_h \text{ and } v_{i,j} = 0 \text{ if } (x_i, y_j) \in \partial\Omega_h\}.$$

For any $u, v \in \mathcal{V}_h$, we define

$$(u, v) = h_1 h_2 \sum_{i=1}^{M_1-1}\sum_{j=1}^{M_2-1} u_{i,j}v_{i,j},$$

and corresponding discrete L^2 norm

$$\|v\| = \sqrt{(v, v)}.$$

In light of Lemma 3.5, the operators $\mathcal{H}_1$ and $\mathcal{H}_2$ are positive definite and self-adjoint. Following the argument in [30], we consider their square roots denoted by Q_1 and Q_2 respectively. Obviously Q_1 and Q_2 are also positive definite and self-adjoint, therefore we can define their inverse operators as Q_1^{-1} and Q_2^{-1} respectively. For those operators, the following results hold.

Lemma 4.8. *For any $v \in \mathcal{V}_h$, it holds that*

$$\frac{1}{3}\|v\| \le \|Q_1Q_2v\| \le \|v\|, \qquad \|v\| \le \|Q_1^{-1}Q_2^{-1}v\| \le 3\|v\|. \tag{4.13}$$

Proof. Noticing the commutativity of the operators Q_1 and Q_2, we have

$$(\mathcal{H}_1\mathcal{H}_2v, v) = (Q_1^2Q_2^2v, v) = (Q_1Q_2v, Q_1Q_2v) = \|Q_1Q_2v\|^2.$$

On the other hand, similar to Lemmas 3.5 and 3.6, it is easy to check that

$$(\mathcal{H}_1\mathcal{H}_2v, v) \le \|v\|^2$$

and

$$\begin{aligned}(\mathcal{H}_1\mathcal{H}_2v, v) &= (\mathcal{H}_2Q_1v, Q_1v) \ge \frac{1}{3}(Q_1v, Q_1v)\\ &= \frac{1}{3}(\mathcal{H}_1v, v) \ge \frac{1}{9}(v, v) = \frac{1}{9}\|v\|^2.\end{aligned}$$

Thus,

$$\frac{1}{3}\|v\| \le \|Q_1Q_2v\| \le \|v\|.$$

Replacing v in the above inequality by $Q_1^{-1}Q_2^{-1}v$, we obtain the second inequality of (4.13). □

Lemma 4.9 ([30]). *For any $v \in \mathcal{V}_h$, it holds that*

$$\begin{aligned}&(\delta_x^\alpha v, v) \le 0, \qquad (v, \delta_x^\alpha v) \le 0,\\ &(\delta_y^\beta v, v) \le 0, \qquad (v, \delta_y^\beta v) \le 0.\end{aligned}$$

With the help of Lemma 4.9, we can prove the following lemmas.

Lemma 4.10. *For any $v \in \mathcal{V}_h$, it holds that*

$$\|(Q_1 - \mu Q_1^{-1}\delta_x^\alpha)(Q_2 - \mu Q_2^{-1}\delta_y^\beta)v\| \ge \|(Q_1Q_2 + \mu^2 Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta)v\|.$$

Proof. According to the definition of the discrete L^2 norm, we have

$$\begin{aligned}
&\|(Q_1-\mu Q_1^{-1}\delta_x^\alpha)(Q_2-\mu Q_2^{-1}\delta_y^\beta)v\|^2\\
&\quad=\|\big(Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta-\mu Q_1Q_2^{-1}\delta_y^\beta-\mu Q_2Q_1^{-1}\delta_x^\alpha\big)v\|^2\\
&\quad=\Big(\big(Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta-\mu Q_1Q_2^{-1}\delta_y^\beta-\mu Q_2Q_1^{-1}\delta_x^\alpha\big)v,\big(Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta\\
&\qquad\qquad-\mu Q_1Q_2^{-1}\delta_y^\beta-\mu Q_2Q_1^{-1}\delta_x^\alpha\big)v\Big)\\
&\quad=\|(Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta)v\|^2+\mu^2\|(Q_1Q_2^{-1}\delta_y^\beta+Q_2Q_1^{-1}\delta_x^\alpha)v\|^2\\
&\qquad-2\mu\big((Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta)v,(Q_1Q_2^{-1}\delta_y^\beta+Q_2Q_1^{-1}\delta_x^\alpha)v\big).
\end{aligned}\tag{4.14}$$

For the third term on the right hand side of (4.14), by Lemma 4.9 and the commutativity of the operators in different spatial direction, we get

$$\begin{aligned}
&\big((Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta)v,(Q_1Q_2^{-1}\delta_y^\beta+Q_2Q_1^{-1}\delta_x^\alpha)v\big)\\
&\quad=\big(Q_1Q_2v,Q_1Q_2^{-1}\delta_y^\beta v\big)+\big(\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta v,Q_1Q_2^{-1}\delta_y^\beta v\big)\\
&\qquad+\big(Q_1Q_2v,Q_2Q_1^{-1}\delta_x^\alpha v\big)+\big(\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta v,Q_2Q_1^{-1}\delta_x^\alpha v\big)\\
&\quad=\big(Q_1v,\delta_y^\beta Q_1v\big)+\mu^2\big(\delta_x^\alpha Q_2^{-1}\delta_y^\beta v,Q_2^{-1}\delta_y^\beta v\big)+\big(Q_2v,\delta_x^\alpha Q_2v\big)+\mu^2\big(\delta_y^\beta Q_1^{-1}\delta_x^\alpha v,Q_1^{-1}\delta_x^\alpha v\big)\le 0.
\end{aligned}$$

Therefore, it follows from (4.14) that

$$\|(Q_1-\mu Q_1^{-1}\delta_x^\alpha)(Q_2-\mu Q_2^{-1}\delta_y^\beta)v\|\ge\|(Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta)v\|.\quad\square$$

Lemma 4.11. *For any $v\in\mathcal{V}_h$, it holds that*

$$\|(Q_1-\mu Q_1^{-1}\delta_x^\alpha)(Q_2-\mu Q_2^{-1}\delta_y^\beta)v\|\ge\frac{1}{3}\|v\|.$$

Proof. It follows from Lemma 4.9 that

$$\begin{aligned}
\|(Q_1-\mu Q_1^{-1}\delta_x^\alpha)v\|^2&=\big((Q_1-\mu Q_1^{-1}\delta_x^\alpha)v,(Q_1-\mu Q_1^{-1}\delta_x^\alpha)v\big)\\
&=\|Q_1v\|^2+\mu^2\|Q_1^{-1}\delta_x^\alpha v\|^2-2\mu(v,\delta_x^\alpha v)\ge\|Q_1v\|^2,
\end{aligned}$$

i.e.,

$$\|(Q_1-\mu Q_1^{-1}\delta_x^\alpha)v\|\ge\|Q_1v\|.$$

Similarly, we have

$$\|(Q_2-\mu Q_2^{-1}\delta_y^\beta)v\|\ge\|Q_2v\|.$$

Thus,

$$\begin{aligned}
\|(Q_1-\mu Q_1^{-1}\delta_x^\alpha)(Q_2-\mu Q_2^{-1}\delta_y^\beta)v\|&\ge\|Q_1(Q_2-\mu Q_2^{-1}\delta_y^\beta)v\|\\
&=\|(Q_2-\mu Q_2^{-1}\delta_y^\beta)Q_1v\|\\
&\ge\|Q_2Q_1v\|\ge\frac{1}{3}\|v\|.
\end{aligned}$$

The last inequality holds by Lemma 4.8. $\square$

Next, we give a prior estimate for the scheme (4.8)–(4.10).

Lemma 4.12. *Suppose $\{v_{i,j}^n\}$ be the solution of*

$$\begin{aligned}
&\mathcal{H}_1\mathcal{H}_2v_{i,j}^n+\mu^2\delta_x^\alpha\delta_y^\beta v_{i,j}^n-\mu\mathcal{H}_2\delta_x^\alpha v_{i,j}^n-\mu\mathcal{H}_1\delta_y^\beta v_{i,j}^n=\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(\mathcal{H}_1\mathcal{H}_2+\mu^2\delta_x^\alpha\delta_y^\beta)v_{i,j}^k\\
&\qquad+a_{n-1}(\mathcal{H}_1\mathcal{H}_2+\mu^2\delta_x^\alpha\delta_y^\beta)v_{i,j}^0+\mu p_{i,j}^n,\quad(x_i,y_j)\in\Omega_h,\ 1\le n\le N,
\end{aligned}\tag{4.15}$$

$$v_{i,j}^n=0,\quad(x_i,y_j)\in\partial\Omega_h,\ 0\le n\le N,\tag{4.16}$$

$$v_{i,j}^0=v_0(x_i,y_j),\quad(x_i,y_j)\in\bar{\Omega}_h,\tag{4.17}$$

where $p_{i,j}^n|_{(x_i,y_j)\in\partial\Omega_h}=0$ *for* $0\le n\le N$. *Then for* $1\le n\le N$,

$$\|v^n\|\le 3\|(Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta)v^0\|+9\Gamma(1-\gamma)T^\gamma\max_{1\le l\le N}\|p^l\|. \tag{4.18}$$

Proof. Eq. (4.15) can be factorized as

$$(\mathcal{H}_1-\mu\delta_x^\alpha)(\mathcal{H}_2-\mu\delta_y^\beta)v_{i,j}^n=\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(\mathcal{H}_1\mathcal{H}_2+\mu^2\delta_x^\alpha\delta_y^\beta)v_{i,j}^k$$
$$+a_{n-1}(\mathcal{H}_1\mathcal{H}_2+\mu^2\delta_x^\alpha\delta_y^\beta)v_{i,j}^0+\mu p_{i,j}^n,\quad (x_i,y_j)\in\Omega_h,\ 1\le n\le N.$$

Multiplying $Q_1^{-1}Q_2^{-1}$ on both sides of the above equation, we obtain

$$(Q_1-\mu Q_1^{-1}\delta_x^\alpha)(Q_2-\mu Q_2^{-1}\delta_y^\beta)v_{i,j}^n=\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})(Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta)v_{i,j}^k$$
$$+a_{n-1}(Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta)v_{i,j}^0+\mu Q_1^{-1}Q_2^{-1}p_{i,j}^n,\quad (x_i,y_j)\in\Omega_h,\ 1\le n\le N.$$

Taking the discrete L^2-norm on both sides and noticing that $(a_{n-k-1}-a_{n-k})$ and a_{n-1} are positive, we have

$$\|(Q_1-\mu Q_1^{-1}\delta_x^\alpha)(Q_2-\mu Q_2^{-1}\delta_y^\beta)v^n\|\le\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|(Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta)v^k\|$$
$$+a_{n-1}\|(Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta)v^0\|+\mu\|Q_1^{-1}Q_2^{-1}p^n\|,\quad 1\le n\le N.$$

By Lemma 4.10, we get

$$\|(Q_1-\mu Q_1^{-1}\delta_x^\alpha)(Q_2-\mu Q_2^{-1}\delta_y^\beta)v^n\|\le\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})\|(Q_1-\mu Q_1^{-1}\delta_x^\alpha)(Q_2-\mu Q_2^{-1}\delta_y^\beta)v^k\|$$
$$+a_{n-1}\|(Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta)v^0\|+\mu\|Q_1^{-1}Q_2^{-1}p^n\|,\quad 1\le n\le N. \tag{4.19}$$

Recall that

$$\mu<\Gamma(1-\gamma)T^\gamma a_{n-1},\quad 1\le n\le N.$$

Denote

$$F=\|(Q_1Q_2+\mu^2Q_1^{-1}\delta_x^\alpha Q_2^{-1}\delta_y^\beta)v^0\|+\Gamma(1-\gamma)T^\gamma\max_{1\le l\le N}\|Q_1^{-1}Q_2^{-1}p^l\|$$

and

$$E^n=\|(Q_1-\mu Q_1^{-1}\delta_x^\alpha)(Q_2-\mu Q_2^{-1}\delta_y^\beta)v^n\|,\quad 1\le n\le N.$$

It follows from (4.19) that

$$E^n\le\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})E^k+a_{n-1}F,\quad 1\le n\le N.$$

Similar to the proof in Lemma 3.7, we can show by mathematical induction that

$$E^n\le F,\quad 1\le n\le N.$$

Obviously, it holds for $n=1$. Assume that

$$E^k\le F,\quad 1\le k\le n-1.$$

Then by Lemma 3.3, we have

$$E^n\le\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})E^k+a_{n-1}F\le\sum_{k=1}^{n-1}(a_{n-k-1}-a_{n-k})F+a_{n-1}F=F.$$

The estimate (4.18) is followed from Lemmas 4.11 and 4.8. □

Applying Lemma 4.12 to the difference scheme (4.8)–(4.10), we immediately obtain the following result.

Theorem 4.3. *The difference scheme (4.8)–(4.10) is unconditionally stable to the initial value* u_0 *and right hand side* f *for all* $0<\gamma<1$ *and* $1<\alpha,\beta\le 2$.

We now consider the convergence of the difference scheme (4.8)–(4.10). Let

$$e_{i,j}^n = U_{i,j}^n - u_{i,j}^n, \quad (x_i, y_j) \in \bar{\Omega}_h,\ 0 \le n \le N.$$

The following theorem holds.

Theorem 4.4. *Assume that $\{U_{i,j}^n\}$ and $\{u_{i,j}^n\}$ are the exact solution of problem (4.1)–(4.3) and difference scheme (4.8)–(4.10), respectively. Then the following estimate*

$$\|e^n\| \le 9c_3\sqrt{(b-a)(d-c)}\,\Gamma(1-\gamma)T^\gamma(\tau^{\min\{2\gamma,2-\gamma\}} + h_1^4 + h_2^4)$$

holds for $1 \le n \le N$ *and for all* $0 < \gamma < 1$, $1 < \alpha, \beta \le 2$.

Proof. By (4.6) and (4.8), the error equations can be written as

$$\begin{aligned}
&\mathcal{H}_1\mathcal{H}_2 e_{i,j}^n + \mu^2\delta_x^\alpha\delta_y^\beta e_{i,j}^n - \mu\mathcal{H}_2\delta_x^\alpha e_{i,j}^n - \mu\mathcal{H}_1\delta_y^\beta e_{i,j}^n = \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(\mathcal{H}_1\mathcal{H}_2 + \mu^2\delta_x^\alpha\delta_y^\beta)e_{i,j}^k \\
&\qquad\qquad + a_{n-1}(\mathcal{H}_1\mathcal{H}_2 + \mu^2\delta_x^\alpha\delta_y^\beta)e_{i,j}^0 + \mu\hat{R}_{i,j}^n, \quad (x_i, y_j) \in \Omega_h,\ 1 \le n \le N, \\
&e_{i,j}^n = 0, \quad (x_i, y_j) \in \partial\Omega_h,\ 1 \le n \le N, \\
&e_{i,j}^0 = 0, \quad (x_i, y_j) \in \bar{\Omega}_h.
\end{aligned}$$

It follows from Lemma 4.12 and the estimate (4.7) that

$$\|e^n\| \le 9\Gamma(1-\gamma)T^\gamma \max_{1\le l\le N}\|\hat{R}^l\| \le 9c_3\sqrt{(b-a)(d-c)}\,\Gamma(1-\gamma)T^\gamma(\tau^{\min\{2\gamma,2-\gamma\}} + h_1^4 + h_2^4), \quad 1 \le n \le N. \quad \square$$

5. Numerical experiments

In this section, numerical examples are given to demonstrate the efficiency of the proposed schemes.

Example 1. Consider the following equation

$$\begin{aligned}
&{}_0^C\mathcal{D}_t^\gamma u(x,t) = {}_0\mathcal{D}_x^\alpha u(x,t) + f(x,t), \quad (x,t) \in (0,1) \times (0,1], \qquad (5.1)\\
&u(0,t) = u(1,t) = 0, \quad t \in (0,1], \\
&u(x,0) = 0, \quad x \in [0,1],
\end{aligned}$$

where the source term

$$f(x,t) = \Gamma(2+\gamma)x^4(1-x)t - \left(\frac{\Gamma(5)}{\Gamma(5-\alpha)}x^{4-\alpha} - \frac{\Gamma(6)}{\Gamma(6-\alpha)}x^{5-\alpha}\right)t^{1+\gamma}.$$

The exact solution of this problem is

$$u(x,t) = x^4(1-x)t^{1+\gamma}.$$

This is a one-dimensional time–space fractional diffusion equation and we solve it by the numerical scheme (3.8)–(3.10). In the test, we compute the maximum norm error of the numerical solution at the last time step by

$$e(\tau, h) = \max_{1\le i\le M-1}|u(x_i, t_N) - u_i^N|,$$

where $u(x_i, t_N)$ represents the exact solution and u_i^N is the numerical solution with the mesh step sizes h and τ at the grid point (x_i, t_N). We first fix the time step $\tau = 1/10^5$ sufficiently small and test the convergence order in spatial direction by letting the spatial step h vary from 1/4 to 1/64. Table 1 presents the numerical results for a variety of α and γ. The "order" in this table is calculated by order $= \log_2(e(\tau, 2h)/e(\tau, h))$. From Table 1, we can observe the fourth order convergence rate in the spatial direction which is consistent with our theoretical analysis.

Next, we fix the spatial step size small enough, say $h = 1/10^3$ and vary the time step τ from 1/20 to 1/320. Table 2 lists the maximum norm errors at time $t = 1$ and convergence orders in temporal direction for a variety of α and γ. The convergence order for this test is calculated by order $= \log_2(e(2\tau, h)/e(\tau, h))$. Evidently, the numerical convergence order in the temporal direction is $\mathcal{O}(\tau^{2-\gamma})$, as in Theorem 3.2.

The following example is a two-dimensional time–space fractional diffusion equation.

Table 1
The maximum errors at time $t = 1$ and convergence orders in spatial direction for Example 1, with $\tau = 1/10^5$.

α	h	$\gamma = 0.75$		$\gamma = 0.5$		$\gamma = 0.25$	
		$e(\tau, h)$	Order	$e(\tau, h)$	Order	$e(\tau, h)$	Order
1.8	1/4	7.1062E−04	*	7.3336E−04	*	7.4895E−04	*
	1/8	4.3563E−05	4.0279	4.4497E−05	4.0428	4.5138E−05	4.0524
	1/16	2.5617E−06	4.0879	2.5970E−06	4.0988	2.6215E−06	4.1059
	1/32	1.5955E−07	4.0050	1.6189E−07	4.0038	1.6420E−07	3.9969
	1/64	1.0655E−08	3.9044	1.0253E−08	3.9809	1.0377E−08	3.9840
1.5	1/4	1.6118E−03	*	1.7128E−03	*	1.7820E−03	*
	1/8	1.0021E−04	4.0075	1.0506E−04	4.0271	1.0837E−04	4.0394
	1/16	5.9226E−06	4.0807	6.1456E−06	4.0955	6.2993E−06	4.1047
	1/32	3.4513E−07	4.1010	3.5425E−07	4.1167	3.6119E−07	4.1244
	1/64	2.0756E−08	4.0555	2.0487E−08	4.1120	2.0783E−08	4.1193
1.2	1/4	2.1508E−03	*	2.4747E−03	*	2.7121E−03	*
	1/8	1.3696E−04	3.9731	1.5279E−04	4.0176	1.6385E−04	4.0490
	1/16	8.4106E−06	4.0254	9.2364E−06	4.0481	9.8085E−06	4.0622
	1/32	5.1192E−07	4.0382	5.5258E−07	4.0631	5.8390E−07	4.0702
	1/64	3.4757E−08	3.8805	3.2853E−08	4.0721	3.4485E−08	4.0817

Table 2
The maximum errors at time $t = 1$ and convergence orders in temporal direction for Example 1, with $h = 1/10^3$.

γ	τ	$\alpha = 1.8$		$\alpha = 1.5$		$\alpha = 1.2$	
		$e(\tau, h)$	Order	$e(\tau, h)$	Order	$e(\tau, h)$	Order
0.75	1/20	7.7499E−05	*	1.3486E−04	*	2.3375E−04	*
	1/40	3.2580E−05	1.2502	5.6747E−05	1.2489	9.8473E−05	1.2471
	1/80	1.3699E−05	1.2500	2.3871E−05	1.2493	4.1451E−05	1.2483
	1/160	5.7599E−06	1.2499	1.0040E−05	1.2495	1.7440E−05	1.2490
	1/320	2.4219E−06	1.2499	4.2221E−06	1.2497	7.3359E−06	1.2494
0.5	1/20	1.3740E−05	*	2.3787E−05	*	4.0953E−05	*
	1/40	4.8685E−06	1.4968	8.4424E−06	1.4944	1.4566E−05	1.4913
	1/80	1.7243E−06	1.4975	2.9936E−06	1.4958	5.1730E−06	1.4936
	1/160	6.1044E−07	1.4981	1.0607E−06	1.4969	1.8348E−06	1.4953
	1/320	2.1603E−07	1.4986	3.7559E−07	1.4978	6.5022E−07	1.4967
0.25	1/20	1.5510E−06	*	2.6467E−06	*	4.4742E−06	*
	1/40	4.6983E−07	1.7230	8.0391E−07	1.7191	1.3636E−06	1.7142
	1/80	1.4171E−07	1.7291	2.4305E−07	1.7258	4.1348E−07	1.7216
	1/160	4.2602E−08	1.7340	7.3213E−08	1.7311	1.2487E−07	1.7274
	1/320	1.2775E−08	1.7377	2.1991E−08	1.7352	3.7588E−08	1.7320

Example 2. Consider the equation

$$\begin{aligned} &{}_0^C\mathcal{D}_t^{\gamma} u(x,y,t) = {}_0\mathcal{D}_x^{\alpha} u(x,y,t) + 2\,{}_x\mathcal{D}_2^{\alpha} u(x,y,t) + {}_0\mathcal{D}_y^{\beta} u(x,y,t) + 2\,{}_y\mathcal{D}_2^{\beta} u(x,y,t) \\ &\qquad + f(x,y,t), \quad (x,y) \in \Omega = (0,2)\times(0,2),\ t \in (0,1], \\ &u(x,y,t) = 0, \quad (x,y) \in \partial\Omega,\ t \in (0,1], \\ &u(x,y,0) = 0, \quad (x,y) \in \bar{\Omega}, \end{aligned} \tag{5.2}$$

where the source term

$$\begin{aligned} f(x,y,t) = {} & \frac{\Gamma(3+\gamma)}{2} x^4(2-x)^4 y^4(2-y)^4 t^2 - \left\{ \frac{16\Gamma(5)}{\Gamma(5-\alpha)} \left[x^{4-\alpha} + 2(2-x)^{4-\alpha}\right] \right. \\ & - \frac{32\Gamma(6)}{\Gamma(6-\alpha)} \left[x^{5-\alpha} + 2(2-x)^{5-\alpha}\right] + \frac{24\Gamma(7)}{\Gamma(7-\alpha)} \left[x^{6-\alpha} + 2(2-x)^{6-\alpha}\right] \\ & \left. - \frac{8\Gamma(8)}{\Gamma(8-\alpha)} \left[x^{7-\alpha} + 2(2-x)^{7-\alpha}\right] + \frac{\Gamma(9)}{\Gamma(9-\alpha)} \left[x^{8-\alpha} + 2(2-x)^{8-\alpha}\right] \right\} y^4(2-y)^4 t^{2+\gamma} \end{aligned}$$

Table 3
The maximum errors at time $t = 1$ and convergence orders in spatial direction for Example 2, with $\tau = 1/(2 \cdot 10^5)$.

(α, β)	h	$\gamma = 1/2$		$\gamma = 2/3$		$\gamma = 3/4$	
		$\hat{e}(\tau, h_1, h_2)$	Order	$\hat{e}(\tau, h_1, h_2)$	Order	$\hat{e}(\tau, h_1, h_2)$	Order
(1.8, 1.6)	2/4	6.2474E−02	*	6.2419E−02	*	6.2366E−02	*
	2/8	5.8221E−03	3.4236	5.7724E−03	3.4348	5.7390E−03	3.4419
	2/16	4.1985E−04	3.7936	4.2221E−04	3.7731	4.1979E−04	3.7731
	2/32	2.4257E−05	4.1134	2.8362E−05	3.8959	2.8308E−05	3.8904
(1.9, 1.2)	2/4	6.0875E−02	*	6.0106E−02	*	5.9651E−02	*
	2/8	6.8311E−03	3.1557	6.7654E−03	3.1513	6.7261E−03	3.1487
	2/16	4.7584E−04	3.8436	4.7187E−04	3.8417	4.6902E−04	3.8421
	2/32	3.2362E−05	3.8781	3.1957E−05	3.8842	3.1724E−05	3.8860
(1.3, 1.4)	2/4	5.9849E−02	*	5.8255E−02	*	5.7311E−02	*
	2/8	5.1187E−03	3.5475	5.0325E−03	3.5330	4.9808E−03	3.5244
	2/16	3.7483E−04	3.7715	3.6861E−04	3.7711	3.6435E−04	3.7730
	2/32	2.9111E−05	3.6866	2.6066E−05	3.8219	2.6054E−05	3.8058

$$
\begin{aligned}
&-\left\{\frac{16\Gamma(5)}{\Gamma(5-\beta)}\left[y^{4-\beta}+2(2-y)^{4-\beta}\right]-\frac{32\Gamma(6)}{\Gamma(6-\beta)}\left[y^{5-\beta}+2(2-y)^{5-\beta}\right]\right.\\
&+\frac{24\Gamma(7)}{\Gamma(7-\beta)}\left[y^{6-\beta}+2(2-y)^{6-\beta}\right]-\frac{8\Gamma(8)}{\Gamma(8-\beta)}\left[y^{7-\beta}+2(2-y)^{7-\beta}\right]\\
&\left.+\frac{\Gamma(9)}{\Gamma(9-\beta)}\left[y^{8-\beta}+2(2-y)^{8-\beta}\right]\right\}x^4(2-x)^4t^{2+\gamma}.
\end{aligned}
$$

The exact solution of this problem is

$$u(x, y, t) = x^4(2-x)^4y^4(2-y)^4t^{2+\gamma}.$$

We solve this equation by the ADI scheme (4.11)–(4.12) with boundary and initial conditions of (4.9) and (4.10). For simplicity, we let $M_1 = M_2 = M$ (or equivalently $h_1 = h_2 = h$) and compute the maximum norm errors of the numerical solution by

$$\hat{e}(\tau, h_1, h_2) = \max_{1\le i,j\le M-1} |u(x_i, y_j, t_N) - u_{i,j}^N|,$$

where $u(x_i, y_j, t_N)$ refers to the exact solution and $u_{i,j}^N$ is the numerical solution with the mesh step sizes h and τ at the grid point (x_i, y_j, t_N). Similar to the one dimensional case, we first fix the time step size $\tau = 1/(2\cdot 10^5)$ sufficiently small and vary the spatial step h from 2/4 to 2/32 to test the spatial convergence rate, and then fix $h = 2/10^3$ sufficiently small and vary τ from 1/32 to 1/512 to verify the temporal convergence rate. The maximum errors and convergence orders in the temporal and spatial directions are reported in Tables 3 and 4, respectively. The numerical results again meet our expectations. We note that for the case with $\gamma = 3/4$ in Table 4, the convergence order in time direction seems to be 2γ which is higher than the theoretical result $\min\{2\gamma, 2-\gamma\}$. We have tested several values of γ with $\gamma > 2/3$ and examples, and observed the similar results. Maybe the theoretical analysis does not provide the best convergence order. We would investigate this point in the future.

6. Concluding remarks

In this paper, we have derived high order finite difference schemes for one- and two-dimensional time–space fractional sub-diffusion equations. The spatial fractional derivatives are discretized by the fourth-order quasi-compact difference scheme [30] and temporal fractional derivatives are approximated by the $L1$ formula. For the two-dimensional case, we have also established an ADI scheme based on $L1$ approximation. By the energy method, we have showed that the proposed schemes are unconditionally stable and convergent. Numerical examples have been provided to illustrate the effectiveness and accuracy of the method.

We point out that some second-order finite difference schemes have also been designed most recently for time fractional diffusion equations [48,14,49]. Vong et al. [50] have employed the discretization formula proposed in [48] and established a finite difference scheme with second-order in time and fourth-order in space for (4.1)–(4.3). Nevertheless, the ADI strategy is not considered in [50]. How to develop an ADI scheme based on those second-order approximation for the temporal derivative and fourth-order quasi-compact difference approximation for the spatial derivative to (4.1)–(4.3) would be our future work. In addition, we would also consider to extend these finite difference schemes to time–space fractional diffusion-wave equations.

Table 4
The maximum errors at time $t = 1$ and convergence orders in temporal direction for Example 2, with $h = 2/10^3$.

γ	τ	$(\alpha, \beta) = (1.8, 1.6)$		$(\alpha, \beta) = (1.9, 1.2)$		$(\alpha, \beta) = (1.3, 1.4)$	
		$\hat{e}(\tau, h_1, h_2)$	Order	$\hat{e}(\tau, h_1, h_2)$	Order	$\hat{e}(\tau, h_1, h_2)$	Order
1/2	1/32	1.4013E−01	*	8.4130E−02	*	7.2981E−02	*
	1/64	7.8195E−02	0.8416	4.4439E−02	0.9208	3.8557E−02	0.9205
	1/128	4.1699E−02	0.9071	2.2858E−02	0.9591	1.9852E−02	0.9577
	1/256	2.1601E−02	0.9490	1.1595E−02	0.9792	1.0079E−02	0.9780
	1/512	1.1003E−02	0.9731	5.8403E−03	0.9894	5.0792E−03	0.9886
2/3	1/32	6.1033E−02	*	3.3530E−02	*	2.8499E−02	*
	1/64	2.5818E−02	1.2412	1.3685E−02	1.2929	1.1643E−02	1.2915
	1/128	1.0528E−02	1.2941	5.4920E−03	1.3172	4.6747E−03	1.3165
	1/256	4.2247E−03	1.3173	2.1892E−03	1.3269	1.8637E−03	1.3267
	1/512	1.6840E−03	1.3269	8.7029E−04	1.3308	7.4096E−04	1.3307
3/4	1/32	3.9396E−02	*	2.0679E−02	*	1.6995E−02	*
	1/64	1.4529E−02	1.4391	7.3827E−03	1.4859	6.0112E−03	1.4993
	1/128	5.1959E−03	1.4835	2.5929E−03	1.5096	2.0851E−03	1.5276
	1/256	1.8348E−03	1.5018	9.0318E−04	1.5215	7.1482E−04	1.5444
	1/512	6.4417E−04	1.5101	3.1282E−04	1.5297	2.4265E−04	1.5587

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